LHS Mathematics Department
Honors Pre-Calculus
Final Exam 2002 Answers

Year	Population
1950	4,691,000
1960	5,149,000
1970	5,689,000
1980	5,737,000
1990	6,016,000
2000	6,349,000

Part A. Short Problems

1. The table at the right gives the population of Massachusetts over the past several decades. Using an exponential model, predict the population of Massachusetts in the year 2015.

Try an exponential regression on the data set:

0	4691
10	5149
20	5689
30	5737
40	6016
50	6349

$p(x) = 4839.135(1.0057)^x$

$p(65) = 7000.8572$

So, this model predicts the 2015 population to be approximately 7,001,000.

Answer to question 1: The population of Mass. in 2015 is predicted to be about 7,001,000.

2. A vending machine operator sells 22,000 cans of soda monthly at a price of 60 cents each. She believes that for each 5 cent increase in price, her monthly sales will drop by 2000 cans. What should her selling price be if she wishes to achieve the maximum possible revenue?

Let n = number of 5 cent price increases.

Revenue = (cost per can)(number of cans sold)

$R(n) = (.60 + .05n)(22000 - 2000n)$

If you consider the domain to be $n \geq 0$, then this Revenue function has a maximum at $n = 0$, meaning that it's best to keep the price at 60 cents.

If you include also include negative values for n (in other words, if you assume that price decreases lead to a comparable amount of increased sales), there is a maximum of $n = -1/2$. If it is possible to have selling prices in between 5-cent increments, this would give an optimal selling price of 57.5 cents. If the price must be a multiple of 5 cents, then 55 cents and 60 cents are the best possible selling prices.

Answer to question 2: Answers of 60, 57.5, or 55 cents were accepted.

3. Find the cube roots of the complex number $(1 - i)$. You may express your answer either in rectangular form or in polar form.

In polar form, $(1-i) = \sqrt{2}cis(\frac{-\pi}{4})$. DeMoivre's Theorem about roots of complex numbers in polar form gives the answers below.

Answer to question 3: The cube roots are $(\sqrt{2})^{1/3} cis(\frac{-\pi/4 + 2\pi k}{3})$, for $k = 0, 1, 2$.

4. Write parametric equations for the ray that meets the following criteria:

- the ray's terminal point is (3, –4)
- the ray must pass through the point (–2, 5)
- $t = 0$ must correspond to the point (3, –4)

Be sure to state the domain of t values.

Use (x, y) = (initial point) + t(velocity vector), where the initial point is the given (3, –4), and the velocity vector is <–2, 5> – <3, –4> = <–5, 9>. This gives $(x, y) = (3, -4) + t\langle -5, 9\rangle$. Breaking the vector equation into two parametric equations, we get:

Answer to question 4:
parametric equations: $x = 3 - 5t$
$y = -4 + 9t$
domain of t values: $0 \le t < \infty$

5. Given ΔABC with $AC = 10$, $CB = 9$, and $\angle A = 35°$, what is the length of side AB?

We are given SSA information for ΔABC, so we can use the Law of Sines (Ambiguous Case).

$$\frac{\sin B}{AC} = \frac{\sin A}{BC}$$
$$\frac{\sin B}{10} = \frac{\sin 35^\circ}{9}$$
$$\sin B = \frac{10 \sin 35^\circ}{9}$$
$$B = \sin^{-1}\left(\frac{10 \sin 35^\circ}{9}\right)$$

So, angle B is either the acute angle 39.59° or the obtuse angle (180°–39.59°) = 140.41°.

Case 1: If angle B is 39.59°, then angle C is 180° – (39.59° + 35°) = 105.41°. Then we can use the Law of Sines to find AB:

$$\frac{AB}{\sin C} = \frac{BC}{\sin A}$$
$$AB = \frac{(\sin C)(BC)}{\sin A}$$
$$AB = \frac{(\sin 105.41^\circ)(9)}{\sin 35^\circ} = 15.13.$$

Case 2: If angle B is 140.41°, then angle C is 180° – (140.41° + 35°) = 4.59°. Then we can use the Law of Sines to find AB:

$$\frac{AB}{\sin C} = \frac{BC}{\sin A}$$
$$AB = \frac{(\sin C)(BC)}{\sin A}$$
$$AB = \frac{(\sin 4.59^\circ)(9)}{\sin 35^\circ} = 1.26.$$

Answer to question 5: AB = 15.13 or 1.26

6. Let $P(x) = 6x^4 - x^3 - 6x^2 - x - 12$.

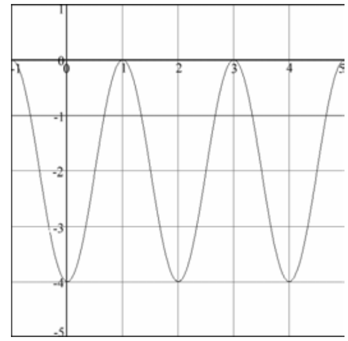
According to the Rational Zeros Theorem, what are the ***possible*** rational zeros of $P(x)$?

$$\frac{\text{factors of } -12}{\text{factors of } 6} = \frac{\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12}{\pm 1, \pm 2, \pm 3, \pm 6}.$$
This gives the list of possible zeros shown below.

Answer to question 6: The possible rational zeros are

$$\pm 1, \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{1}{6}, \pm 2, \pm \frac{2}{3}, \pm 3, \pm \frac{3}{2}, \pm 4, \pm \frac{4}{3}, \pm 6, \pm 12.$$

7. Find a model involving the sine function for the given graph.



Vertically, the amplitude of 2 and there is a shift downward by 2. Modeling just these features would give the formula $(2 \sin x) - 2$.

Horizontally, the period is 2, which gives a coefficient of $2\pi/2 = \pi$. Now the formula is $2 \sin(\pi x) - 2$.

All that remains is to modify the formula to shift the graph horizontally. One possibility is to add a shift by 1/2 unit to the right. This gives the following as a final model:

Answer to question 7: $f(x) = 2 \sin(\pi(x - 1/2)) - 2$

8. Find the angle measure (in radians) between vectors **u** and **v** in the given diagram. [Diagram showed vector **u** from (0, 1.5) to (–1, 0) and vector **v** from (0, 1.5) to (2.5, 0).]

Let θ represent the angle between **u** and **v**.

$$\cos\theta = \frac{u \cdot v}{|u||v|} = \frac{\langle -1,-1.5\rangle \cdot \langle 2.5,-1.5\rangle}{|\langle -1,-1.5\rangle||\langle 2.5,-1.5\rangle|} \approx \frac{-0.25}{1.601 \cdot 2.916} \approx -0.054.$$

Use inverse cosine to get $\theta \approx 1.62$ radians or 93°.

Another approach: Use Law of Cosines on the side lengths of the triangle.

Another approach: Find acute angles of two right triangles, then add.

Answer to question 8: The angle measure is 1.62 radians.

Part B. Medium Problems

9. **a.** How many ways can you elect a 4-person committee from a group of 10 people?

Order does not matter, so ${}_{10}C_4$ or 210.

b. How many ways can you elect a president, a vice-president, a treasurer, and a secretary from a group of 10 people? (The jobs must be held by 4 different people.)

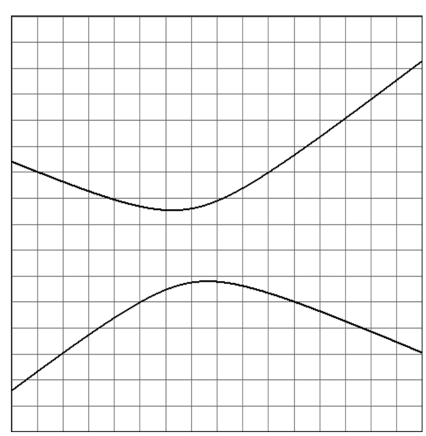
Order does matter, so $10 \cdot 9 \cdot 8 \cdot 7 = {}_{10}P_4 = 5040$.

c. If a 4-person committee is chosen at random from a group of 10 people, and if Amy and Katie are two of the 10 people, what is the probability that Amy and Katie are both on the committee?

$$\frac{\text{number of committees containing both Amy and Katie}}{\text{total number of committees}} = \frac{{}_8C_2}{{}_{10}C_4} \approx .133 \text{ or } 13.3\% \text{ chance}$$

10. Using your calculator, graph this equation on the grid: $-x^2 - xy + 3y^2 - 3x + 4y = 6$.

One way is to group the y^2, y, and constant terms and then use the quadratic formula to solve for y.



$$-x^2 - xy + 3y^2 - 3x + 4y = 6$$
$$3y^2 + 4y - xy - x^2 - 3x - 6 = 0$$
$$(3)y^2 + (4 - x)y + (-x^2 - 3x - 6) = 0$$
$$y = \frac{-(4-x) \pm \sqrt{(4-x)^2 - 4(3)(-x^2 - 3x - 6)}}{2(3)}$$

11. Show the algebraic steps to find **all** solutions to the trigonometric equation

$$\sin^2 x - \sin x = \cos^2 x .$$

$\sin^2 x - \sin x = \cos^2 x$

$\sin^2 x - \sin x = 1 - \sin^2 x$

$2\sin^2 x - \sin x - 1 = 0$

$2w^2 - w - 1 = 0$

$w = \dfrac{1 \pm \sqrt{1 - 4(2)(-1)}}{2(2)} = 1, -1/2$

$\sin x = 1 \text{ or } \sin x = -1/2$

$x = \{\frac{-\pi}{6}, \frac{-5\pi}{6}, \frac{\pi}{2}\} + 2\pi n$

12. Answer the following questions about the series $\sum_{n=4}^{80}(3n - 16)$.

a. Fill in the first three terms and the last three terms of the series.

(-4) + (-1) + (2) +. . . + (218) + (221) + (224)

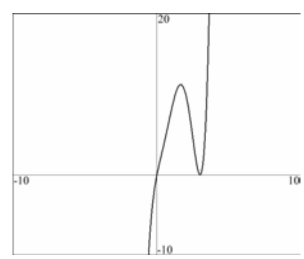
b. Is the series an *arithmetic series*, a *geometric series*, or neither of these?

There is a common difference of 3, so it is arithmetic.

c. Find the sum of the series, or explain why the series does not have a sum.

All finite arithmetic series have finite sums, so $n(\frac{a_1 + a_n}{2}) = 77(\frac{-4+224}{2}) = 8470$.

13. You are given the following information about $P(x)$:



- $P(x)$ is a polynomial with real coefficients.
- The graph of $P(x)$ for real numbers x is shown.
- The only real zeros of $P(x)$ are 0 and 3.
- $P(i) = 0$.

Write a possible formula for $P(x)$. You may leave it in either standard form or factored form. Explain how you get your formula.

From the given information we know there is a zero at x = 0, and from the graph we know the zero at x = 3 is a double root. Since all the coefficients are real, we know the imaginary root's conjugate is also a root. That gives us 5 roots, which matches with the picture at right.

$P(x) = (x - 0)(x - 3)(x - 3)(x - i)(x + i)$

$P(x) = x(x - 3)^2(x^2 + 1)$

This a *a* polynomial with the appropriate zeros. Our polynomial could by scaled by any real number to stretch it vertically. Checking our polynomial with the graph of the polynomial at right, we see that it's current scaling by 1 is reasonable.

Note: Mr. Kresser's class had a different problem for problem **14**.

14. Find the partial fraction decomposition of $\dfrac{9x^2-10x-6}{x^3-x^2-6x}$.

$$\frac{9x^2-10x-6}{x^3-x^2-6x}=\frac{9x^2-10x-6}{x(x-3)(x+2)}=\frac{A}{x}+\frac{B}{x-3}+\frac{C}{x+2}$$

Then, multiply both sides of the equation by $x(x-3)(x+2)$ to get:

$$9x^2-10x-6=A(x-3)(x+2)+B(x)(x+2)+C(x)(x-3)$$
$$9x^2-10x-6=A(x^2-x-6)+B(x^2+2x)+C(x^2-3x)$$
$$9x^2-10x-6=x^2(A+B+C)+x(-A+2B-3C)+(-6A).$$

Equating like coefficients gives the following system of equations:

$$A+B+C=9$$
$$-A+2B-3C=-10$$
$$-6A=-6.$$

We can solve this system using an augmented 3-by-4 matrix:

$$\text{rref}\left(\begin{bmatrix}1 & 1 & 1 & 9\\ -1 & 2 & -3 & -10\\ -6 & 0 & 0 & -6\end{bmatrix}\right)=\begin{bmatrix}1 & 0 & 0 & 1\\ 0 & 1 & 0 & 3\\ 0 & 0 & 1 & 5\end{bmatrix}.$$

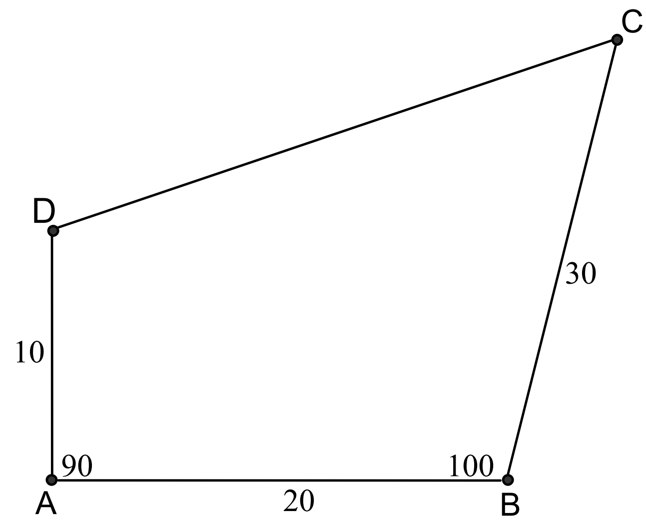
So, $A = 1$, $B = 3$, and $C = 5$. Therefore, the partial fraction decomposition is:

$$\frac{9x^2-10x-6}{x^3-x^2-6x}=\frac{1}{x}+\frac{3}{x-3}+\frac{5}{x+2}.$$

Part C. Long Problems

4 problems, 8 points each, 32 points total

15. Consider the given quadrilateral ABCD.



a. Find the missing side length and angle measures. Write your answers on the diagram.

First find *DB*:

$$(DB)^2 = 10^2 + 20^2$$
$$DB = 10\sqrt{5}$$

Find <*DBA* and <*DBC*:

$$< DBA = \sin^{-1}(\tfrac{10}{10\sqrt{5}}) \approx 26.57^\circ \text{, so}$$
$$< DBC \approx 73.43^\circ$$

Find *DC*:

$$(DC)^2 = 30^2 + (10\sqrt{5})^2 - 2(30)(10\sqrt{5})\cos 73.43^\circ$$
$$DC \approx 31.90$$

Find <*BCD*:

$$\frac{\sin < BCD}{10\sqrt{5}} = \frac{\sin 73.43^\circ}{31.90}$$
$$< BCD \approx 42.21^\circ$$

Find <*CDA*:

$$< CDA = 360^\circ - 100^\circ - 90^\circ - 42.21^\circ = 127.79^\circ$$

b. Find the area of ABCD.

$$\text{Area of ABCD} = \text{Area of } \Delta\text{ABD} + \text{Area of } \Delta\text{BCD}$$
$$= .5(10)(20) + .5(10\sqrt{5})(30)\sin 73.43^\circ \approx 421.48$$

c. Let point A lie at the origin of a Cartesian coordinate system with segment AB along the positive *x*-axis. Find the coordinates of a point P, lying on diagonal BD, such that Area of ΔABP = Area of ΔADP.

Let DP = x, so that BP = $10\sqrt{5} - x$

$$.5(10)(x)\sin < ADB = .5(20)(10\sqrt{5} - x)\sin < ABD$$
$$.5(10)(x)\sin 63.43^\circ = .5(20)(10\sqrt{5} - x)\sin 26.57^\circ$$
$$x \approx 11.18$$

Draw a right triangle as shown and use right triangle trig. to find (x,y) coordinate of P.

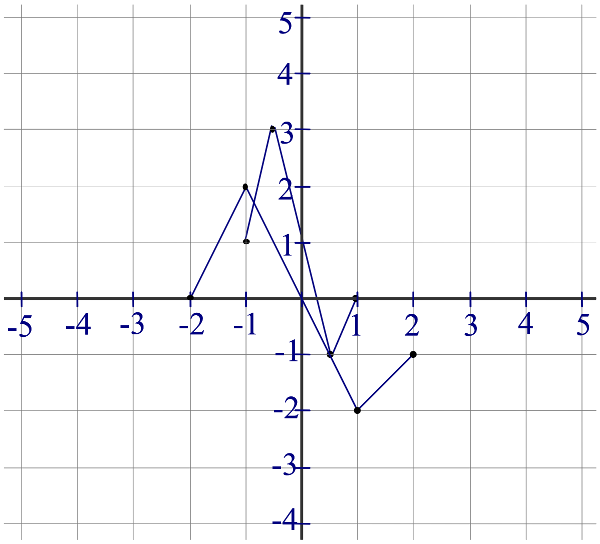
The point is (10.00, 5.00).

Another possible argument: Consider *DP* and *PB* as the bases of the triangles. The heights of the two triangles to these bases are equal. To have equal areas, the bases must also be equal. Thus *DP* = *PB*, making *P* the midpoint of *DB*.

Another possible argument: Δ*ADB* has area 100, so Δ*APB* has area 50. Using *AB* = 20 as the base of Δ*APB*, the height must be 5. This shows *P* must have a *y*-coordinate of 5; then find $x = 10$.

16. Each part of this problem involves transformations.

a. Given the graph of $F(x)$ on the first grid, sketch the graph of $F(2x) + 1$ on the second grid.

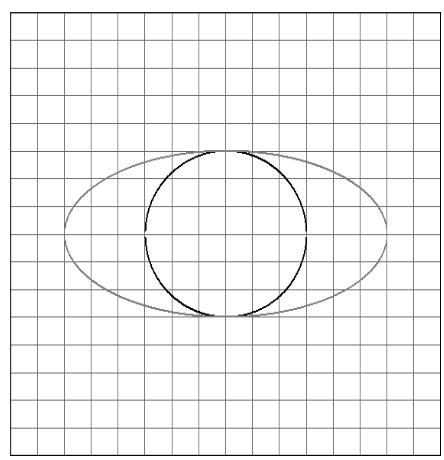


b. Find the periods of the functions $\tan\theta$ and $\tan(k\theta)$.
Use the unit circle definition of tangent to justify your first answer, and use a transformation argument to justify your second answer.

The period of $\tan\theta$ is π because $\tan\theta$ is defined as y/x. The values of this ratio for third quadrant angles are the same as for the corresponding first quadrant angles, because both x and y are opposite in the third quadrant. Similarly, $\tan\theta$ for fourth quadrant angles take the same values as second quadrant angles. Therefore, the values of $\tan\theta$ repeat after an interval of π.

The period of $\tan(k\theta)$ is π/k because the factor of k shrinks the graph horizontally by a factor of k.

c. A circle is described parametrically by $x = 3\cos t$, $y = 3\sin t$. If this circle is stretched horizontally by a factor of 2, an ellipse is obtained. Sketch the circle and the ellipse, then write a pair of parametric equations describing the ellipse.



For the ellipse, just double the x coordinate:

$$x = 6\cos t,\ y = 3\sin t.$$

17. For all parts of this problem, suppose functions $f(x)$ and $g(x)$ are **inverses** of each other.

a. If point (2, 9) lies on the graph of $g(x)$, what can be said about the graph of $f(x)$?

Point (9, 2) lies on the graph of $f(x)$.

b. Re-express $f(g(f(8)))$ in the simplest possible form.

$f(8)$

c. Given that $f(x) = \log_3 x$, identify $g(x)$, and find $g(-4)$.

$g(x) = 3^x$; $g(-4) = 3^{-4} = 1/81$.

d. Given that $f(x) = \log_3 x$, re-express $f(5a) - 4f(b) + 2$ in the form $f(\ \cdots\ )$.

$$f(5a) - 4f(b) + 2$$
$$\log_3(5a) - 4\log_3(b) + \log_3 9$$
$$\log_3(5a) - \log_3(b^4) + \log_3 9$$
$$\log_3(\tfrac{5a}{b^4}) + \log_3 9$$
$$\log_3(\tfrac{45a}{b^4})$$
$$f(\tfrac{45a}{b^4})$$

18. Do the following for the function $f(x) = \dfrac{2x^2 - 6x - 20}{(x-5)^2(x+2)}$.

a. Find the $(x,\ y)$ coordinates of any removable discontinuities of the graph of $f(x)$.

The rational function factors into $\dfrac{2(x-5)(x+2)}{(x-5)^2(x+2)}$. So, there is a removable discontinuity at $x = -2$.

The coordinates of that removable discontinuity are: $(-2,\ f(-2)) = \left(-2,\ \dfrac{2}{(-2-5)}\right) = \left(-2,\ \dfrac{2}{-7}\right)$

b. State the domain and the range of $f(x)$.

The domain of $f(x)$ is $(-\infty,-2) \cup (-2,5) \cup (5,\infty)$.

The range of $f(x)$ is $\left(-\infty, \dfrac{-2}{7}\right) \cup \left(\dfrac{-2}{7}, 0\right) \cup (0, \infty)$

c. Determine each of the following limits. If the limit is not a finite number, chose one of the following answers: $\infty, -\infty$, or "does not exist."

$$\lim_{x\to 5^-} \frac{2(x-5)(x+2)}{(x-5)^2(x+2)} = -\infty.$$
$$\lim_{x\to -2} \frac{2(x-5)(x+2)}{(x-5)^2(x+2)} = \lim_{x\to -2} \frac{2(x-5)}{(x-5)^2} = \frac{-2}{7}.$$
$$\lim_{x\to 0} \frac{2(x-5)(x+2)}{(x-5)^2(x+2)} = \frac{2(-5)(2)}{(-5)^2(2)} = \frac{-2}{5}.$$
$$\lim_{x\to -\infty} \frac{2(x-5)(x+2)}{(x-5)^2(x+2)} = 0.$$

d. For what values of x is $f(x)$ increasing?

$f(x)$ is never increasing.