Name _____

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Class block (circle): A B C D E F G H

Lexington High School
Mathematics Department
Honors Pre-Calculus
Final Exam 2003 Solutions

This is a 90-minute exam, but you will be allowed to work for up to 120 minutes.

The exam has 3 parts. Directions for each part appear below.

In total, there are 72 points that you can earn. A letter grade scale will be set by the course faculty after the tests have been graded.

Part A. Short Problems

8 questions, 2 points each, 16 points total

You must write your answers in the answer boxes.

If your answer is correct, you will receive full credit. Showing work is not required.

If your answer is incorrect, you may receive half credit if you have shown some correct work.

Part B. Medium Problems

6 problems, 4 points each, 24 points total

Write a complete, clearly explained solution to each problem. Partial credit will be given.

Part C. Long Problems

4 problems, 8 points each, 32 points total

Write a complete, clearly explained solution to each problem. Partial credit will be given.

Part A. Short Problems

8 problems, 2 points each, 16 points total

1. Find the solution to this system of equations.

$$4A + 7B + 2C = -7$$
$$3A + 5B - 9C = 26$$
$$6A - 9B - 8C = 3$$

Solution

The easiest method would be to solve the following matrix equation:

$$\begin{bmatrix} 4 & 7 & 2 \\ 3 & 5 & -9 \\ 6 & -9 & -8 \end{bmatrix} \cdot \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} -7 \\ 26 \\ 3 \end{bmatrix} \quad \text{so that} \cdot \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 4 & 7 & 2 \\ 3 & 5 & -9 \\ 6 & -9 & -8 \end{bmatrix}^{-1} \cdot \begin{bmatrix} -7 \\ 26 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$$

$(A,B,C) = (-2, 1, -3)$.

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2. Suppose $f(x) = 5\sqrt{x}$. Find $f^{-1}(x)$ and state the domain of f^{-1}.

Solution

$x = 5y^{1/2}$

$x/5 = y^{1/2}$

$x^2/25 = y = f^{-1}(x)$

The domain of f^{-1} must be the range of f. f has a range of $[0,\infty)$, so f^{-1} has a domain of $[0,\infty)$.

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3. A linear transformation T maps the point (0, 1) to (3, 4) and the point (1, 0) to (–2, 5). To what point does T map the point (a, b)?

Solution

The linear transformation equations for this map are $x' = -2x + 3y$ and $y' = 5x + 4y$.
Therefore the image of (a,b) is $(-2a + 3b, 5a + 4y)$.

4. For what values of A will the series $\sum_{k=1}^{\infty}(3A)^k$ have a finite value?

Solution

This series is an infinite geometric series. Such series have a finite value when the ratio is between –1 and 1. In this case, the ratio is $3A$, so $-1 < 3A < 1$, and $-\frac{1}{3} < A < \frac{1}{3}$.

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5. Re-express in as simple a form as possible:

$$\log(10m) + \log(100m) + \log(1000m)$$

Solution

$$\begin{array}{llllll} \log(10m) & + \log(100m) & + \log(1000m) & = \\ \log(10) + \log(m) & + \log(100) + \log(m) & + \log(1000) & + \log(m) = \\ 1 + \log(m) & + 2 + \log(m) & + 3 & + \log(m) = & 6 + 3\log(m). \end{array}$$

Note: Mr. Rahman's class had a different problem for #5.

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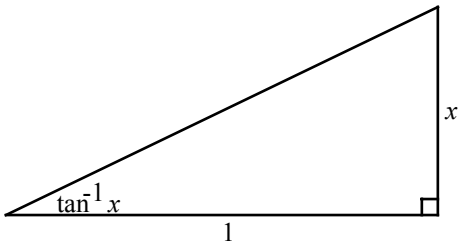
6. Assuming that $x > 0$, find an algebraic expression (not involving trigonometric functions) that is equal to $\sin(\tan^{-1}(x))$.

Solution

Since $x > 0$, $\tan^{-1}(x)$ is a first-quadrant angle, and this diagram can be drawn:



By the Pythagorean Theorem, the length of the hypotenuse is $\sqrt{x^2+1}$. Then,

$$\sin(\tan^{-1} x) = \frac{\text{opp.}}{\text{hyp.}} = \frac{x}{\sqrt{x^2+1}}.$$

7. Rewrite the following equation in the standard form for a hyperbola.

$$9x^2 - 4y^2 - 18x + 16y - 11 = 0$$

Solution

$$9x^2 - 4y^2 - 18x + 16y - 11 = 0$$
$$9(x^2 - 2x + 1) - 4(y^2 - 4y + 4) = 11 + 9 - 16$$
$$9(x-1)^2 - 4(y-2)^2 = 4$$
$$\frac{(x-1)^2}{(4/9)} - \frac{(y-2)^2}{1} = 1 \qquad \text{- or -}$$
$$\frac{(x-1)^2}{(2/3)^2} - \frac{(y-2)^2}{1^2} = 1$$

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8. Find the three cube roots of 8, expressing your answers in polar form.

Solution

One of the cube roots is the real number 2, or in polar form 2cis0°. We also know that all three cube roots are spaced at the vertices of a regular 3-gon (triangle). Therefore, the other roots must be at 2cis120° and 2cis240°.

The cube roots are 2cis0°, 2cis120°, and 2cis240°.

Part B. Medium Problems

6 problems, 4 points each, 24 points total

9. Using an algebraic (not graphical) method, find all solutions in the interval $0 \le x \le 2\pi$ to the equation $\sin(5x) = -\frac{1}{2}$. Show complete work.

Solution

The two locations on the unit circle having a sine value of $-\frac{1}{2}$ are at angles of $\frac{7\pi}{6}$ and $\frac{11\pi}{6}$. The complete set of angles having a sine value of $-\frac{1}{2}$ consists of $\frac{7\pi}{6} + 2\pi n$ and $\frac{11\pi}{6} + 2\pi n$, for all integers n. Thus,

$$5x = \tfrac{7\pi}{6} + 2\pi n \quad \text{or} \quad 5x = \tfrac{11\pi}{6} + 2\pi n$$
$$5x = \tfrac{(7+12n)\pi}{6} \quad \text{or} \quad 5x = \tfrac{(11+12n)\pi}{6}$$
$$x = \tfrac{(7+12n)\pi}{30} \quad \text{or} \quad x = \tfrac{(11+12n)\pi}{30}$$

In the requested interval $[0, 2\pi]$, the solutions are $\frac{7\pi}{30}, \frac{11\pi}{30}, \frac{19\pi}{30}, \frac{23\pi}{30}, \frac{31\pi}{30}, \frac{35\pi}{30}, \frac{43\pi}{30}, \frac{47\pi}{30}, \frac{55\pi}{30}, \frac{59\pi}{30}$.

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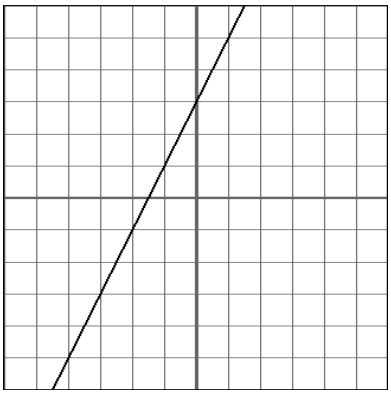
10. Suppose that $\ln(f(x))$ equals the linear function whose graph is shown on the grid.



graphing window: [–6, 6] by [–6, 6]

a. What type of function is $f(x)$?

Solution exponential

b. Find a formula for $f(x)$.

Solution

$\ln(f(x)) = 2x + 3$, so $f(x) = e^{2x+3}$.

Note: Mr. Rahman's class had a different problem for #5.

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11. Suppose that point $P = (2, 3)$ is rotated 120° counterclockwise around the origin. Let point Q be the image of P under this rotation. Find the coordinates of Q.

Solution

Use multiplication by the transformation matrix for the 120° rotation:

$$[2 \quad 3] \cdot \begin{bmatrix} \cos 120° & \sin 120° \\ -\sin 120° & \cos 120° \end{bmatrix} \approx [-3.598 \quad .2320].$$

So $Q \approx (-3.598 \quad .2320)$.

12. The following problem is about probabilities associated with a standard deck of playing cards. A standard deck contains 13 red diamonds, 13 red hearts, 13 black spades and 13 black clubs.

a. If you are dealt a hand consisting of 5 cards, find the probability of being dealt a hand containing 3 hearts and 2 black cards.

Solution $\dfrac{\binom{13}{3}\cdot\binom{26}{2}}{\binom{52}{5}} \approx .036$

b. If you are dealt 5 cards ***sequentially***, find the probability of being dealt 3 hearts ***followed by*** 2 black cards.

Solution $\dfrac{13}{52}\cdot\dfrac{12}{51}\cdot\dfrac{11}{50}\cdot\dfrac{26}{49}\cdot\dfrac{25}{48} \approx .0036$

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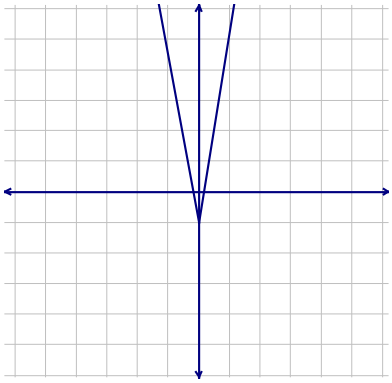
13. Suppose that $f(x) = |x|$. List the transformations needed to transform $f(x)$ into $3(f(2x)) - 1$, and sketch a graph of $3(f(2x)) - 1$.

Solution

1st trans.: Horizontal Shrink by ½

2nd trans.: Vertical Stretch by 3

3rd trans.: Vertical Shift down by 1.



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14. Given $h(x) = x^3 - x + 3$.

a. How many non-real complex zeroes does $h(x)$ have? Explain how you get your answer.

Solution

The Fundamental Theorem of Algebra guarantees that h must have 3 zeroes over the complex numbers. Graphically, h crosses the x – axis once, which implies that h has exactly one real zero. The other two must be complex (in fact, they must be complex conjugates) ∴ There are two non-real complex zeros.

b. Prove that $h(x)$ must have an irrational zero. ***Hint:*** Consider the possible rational zeroes.

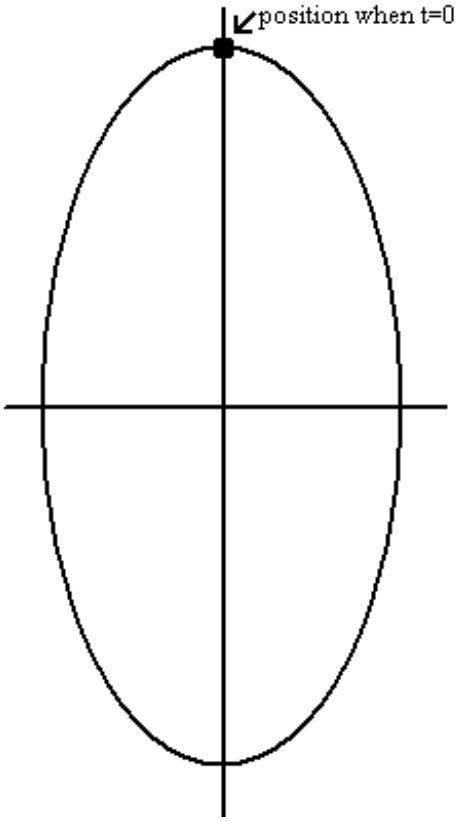
Solution

The possible rational zeros of h are $\pm\frac{3}{1}, \pm\frac{1}{1}$. Both synthetic division and the Remainder Theorem could show that none of these possible rational zeros are zeros of h. Therefore the one real zero of h must be irrational.

Part C. Long Problems

4 problems, 8 points each, 32 points total

15. A planet has an elliptical orbit *with the sun at one of the focal points*. Consider the orbit in the *xy* plane, centered at the origin.



The major axis, along the *y*-axis, has length 200 gigameters. The minor axis, along the *x*-axis, has length 120 gigameters. (1 gigameter = 10^6 km = 10^9 m.)

a. Write a rectangular equation (i.e., equation involving *x* and *y*) describing the ellipse.

Solution

$$\frac{x^2}{60^2}+\frac{y^2}{100^2}=1$$

b. The sun is located at a focal point. Find a possible set of coordinates for the sun, and add the sun to the diagram.

Solution

$60^2 + c^2 = 100^2$, giving $c = \pm 80$. So the sun is either at (0, 80) on the positive *y*-axis or at (0, –80) on the negative *y*-axis.

c. Let *t* represent time measured in days. When $t = 0$, the planet is located as shown in the diagram. The planet orbits counterclockwise, completing an orbit every 2π days. Write a pair of parametric equations modeling the orbiting planet.

Solution

$x(t) = 60 \cos(t + \pi/2)$ $\qquad$ $y(t) = 100 \sin(t + \pi/2)$

which simplify to

$x(t) = -60 \sin t$ $\qquad$ $y(t) = 100 \cos t$

d. Rewrite your equations from part **c** to get formulas for sin *t* and cos *t*.

Solution

$\sin t = \frac{x(t)}{-60}$ $\qquad$ $\cos t = \frac{y(t)}{100}$

e. Perform a calculation that combines your part **d** equations to produce the equation from part **a**.

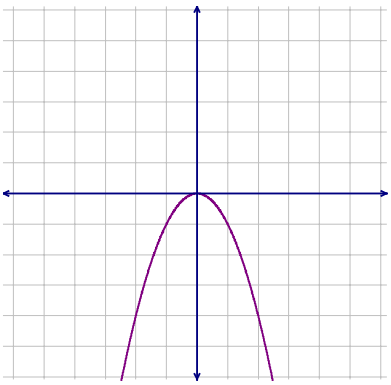
Solution

$\sin^2 t + \cos^2 t = 1$

$\left(\frac{x(t)}{-60}\right)^2 + \left(\frac{y(t)}{100}\right)^2 = 1$

$$\frac{(x(t))^2}{60^2}+\frac{(y(t))^2}{100^2}=1$$

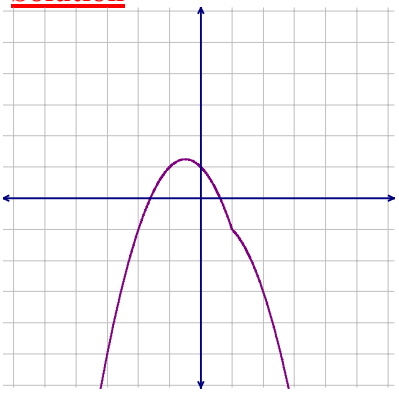
16. Suppose that $f(x) = |x - 1|$, and that $g(x)$ is the function whose graph is shown on the grid.



graphing window: [–6, 6] by [–6, 6]

a. Sketch the graph of the sum function $f + g$.

Solution



b. Write a possible formula for $g(x)$.

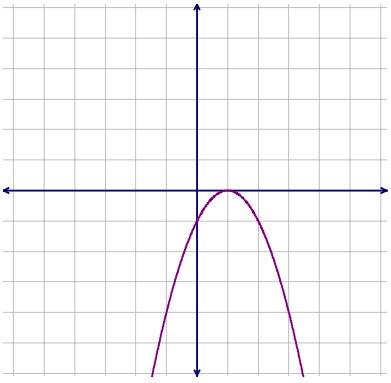
Solution $g(x) = -x^2$

c. Based on your answer to part **b**, write a formula for the quotient function $\frac{g}{f}$, and identify the domain for this function.

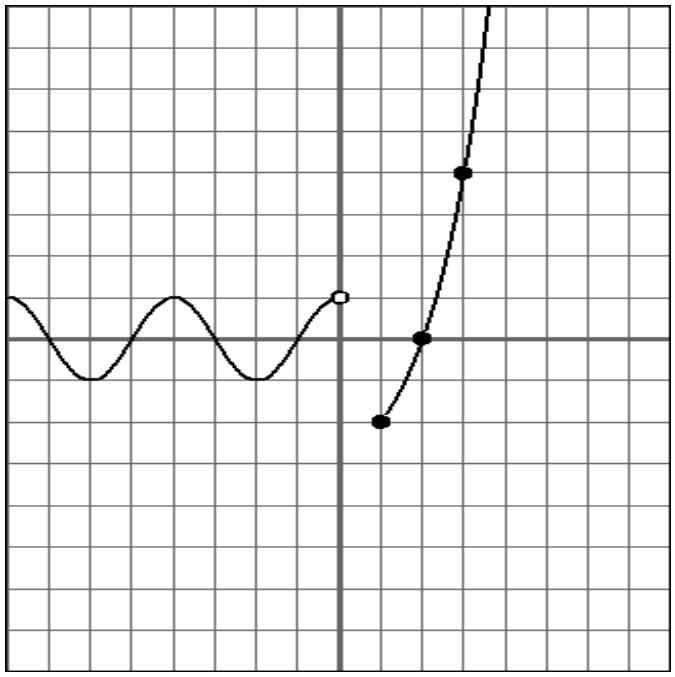
Solution

$\frac{g}{f}(x) = -\frac{x^2}{|x-1|}$, which has a domain of $(-\infty,1) \cup (1,\infty)$.

d. Sketch the graph of the composite function $g(f(x))$.



17. Here is a graph of a function $F(x)$, along with a table showing selected values of the function.



graphing window: [–8, 8] by [–8, 8]

x	$F(x)$
–6	–1
–4	1
–2	–1
0	undef.
1	–2
2	0
3	4

For $-\infty < x < 0$, $F(x)$ is sinusoidal.
For $1 \le x < \infty$, $F(x)$ has the general form $b^x + c$.

a. Identify the domain and range of F.

Solution

domain: $(-\infty,0)\cup[1,\infty)$ *range:* $[-2,\infty)$

b. Write a piecewise definition for F.

Solution

$$F(x)=\begin{cases}\cos(\frac{\pi}{2}x) & (-\infty,0)\\ 2^x-4 & [1,\infty)\end{cases}$$

Each piece of this function must be modeled separately. The sinusoid has a period of 4, and therefore a "b" term of $2\pi/4=\pi/2$. The other piece can be found by using two of the data points and solving the resulting two equation system:

$$b^1+c=-2 \text{ and } b^2+c=0 \quad \Rightarrow b=2 \text{ and } c=-4$$

c. Identify the following limits. If a limit is not a finite number, give one of the following answers: ∞, $-\infty$, or undefined.

Solution

$\lim_{x\to 0^-} F(x) = 1$ $\qquad$ $\lim_{x\to\infty} F(x) = \infty$

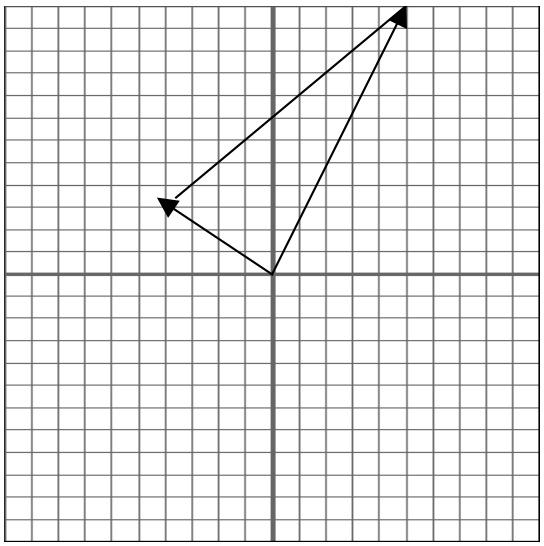
$\lim_{x\to 0} F(x) =$ undefined $\qquad$ $\lim_{x\to -\infty} F(x) =$ undefined

d. Identify the linear function on the interval [0, 1) that would extend F to become a continuous function for all real numbers x.

Solution

A line with slope –3 and passing though the point (0,1). $F(x) = -3x+1$ defined on [0,1).

18. Let $A = (0, 0)$. Consider the vectors $\overrightarrow{AB} = \langle -4, 3 \rangle$ and $\overrightarrow{AC} = \langle 5, 12 \rangle$. Draw them on the given grid.



a. Find the magnitudes of $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

Solution

Magnitude is the equivalent of the hypotenuse of a triangle formed from the components.

Magnitude of $\overrightarrow{AB} = 5$

Magnitude of $\overrightarrow{AC} = 13$

b. Calculate the angle between the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

Solution

$$\cos(\theta) = \frac{\vec{AB} \cdot \vec{AC}}{|AB||AC|}$$

$$\Rightarrow \cos(\theta) = \frac{-4(5) + 3(12)}{5 \cdot 13}$$

$$\Rightarrow \theta = 75.750^\circ$$

c. Consider triangle ABC, where vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ form two of the sides. What is the area of ΔABC?

Solution

$$\text{Area} = \tfrac{1}{2}\text{side}_1 \cdot \text{side}_2 \cdot \sin(\text{angle in between})$$

$$\text{Area} = .5(5)(13)(\sin 75.750) = 31.5$$

d. Draw the angle bisector of angle A. Find the coordinates of the point where this ray intersects side BC.

Solution

Several different methods will work here:
Solution 1: Find the line that has the same slope and intercept as vector BC. Then find the line that describes the angle bisector. Find the intersection of the two lines.
Solution 2: Use a combination of the Law of Sines, Law of Cosines, and right triangle trigonometry.
$(x,y) = (-1.5, 5.5)$