Name Solution Document

Teacher (circle): Kelly Kresser Verner

Class block (circle): B D E G H

Honors Advanced Math
Final Exam 2005 –Solutions

Lexington High School
Mathematics Department

This is a 90-minute exam, but you will be allowed to work for up to 120 minutes.

The exam has 3 parts. Directions for each part appear below.

In total, there are 68 points that you can earn. A letter grade scale will be set by the course faculty after the tests have been graded.

Part A. Short Problems

6 questions, 2 points each, 12 points total

You must write your answers in the answer boxes.

If your answer is correct, you will receive full credit. Showing work is not required.

If your answer is incorrect, you may receive half credit if you have shown some correct work.

A good pace on this part would be to spend around 3 minutes per problem.

Part B. Medium Problems

8 problems, 4 points each, 32 points total

Write a complete, clearly explained solution to each problem. Partial credit will be given.

A good pace on this part would be to spend around 5 minutes per problem.

Part C. Long Problems

3 problems, 8 points each, 24 points total

Write a complete, clearly explained solution to each problem. Partial credit will be given.

A good pace on this part would be to spend around 10 minutes per problem.

Part A. Short Problems

6 problems, 2 points each, 12 points total

1. Functions $f(x) = \cos x$ and $g(x) = |x|$ are given.
Let $h(x) = g(f(x))$, where the domain of $h(x)$ is restricted to $\frac{\pi}{2} \le x \le \pi$.
Find the value of $h^{-1}(\frac{1}{2})$.

Solution:

$h(x) = g(f(x)) = |\cos x|$. Need to find angle $x = h^{-1}(\frac{1}{2})$. Value of x must satisfy two conditions: $|\cos x| = \frac{1}{2}$ and $\frac{\pi}{2} \le x \le \pi$. Since $\cos x$ can't be positive on $\frac{\pi}{2} \le x \le \pi$, $\cos x = -\frac{1}{2}$. Therefore, $x = h^{-1}(\frac{1}{2}) = \frac{2\pi}{3}$

Answer to question 1:

$h^{-1}(\frac{1}{2}) = \frac{2\pi}{3}$

2. Find values of A and B such that $\frac{2x-5}{(x+1)(x+5)} = \frac{A}{(x+1)} + \frac{B}{(x+5)}$.

Solution: $\frac{2x-5}{(x+1)(x+5)} = \frac{A}{(x+1)} + \frac{B}{(x+5)} \Rightarrow \frac{2x-5}{(x+1)(x+5)} = \frac{A(x+5)}{(x+1)(x+5)} + \frac{B(x+1)}{(x+5)(x+1}$

$$2x - 5 = (A+B)x + (5A+B)$$

This yields the linear system: $\begin{cases} A+B=2 \\ 5A+B=-5 \end{cases} \Rightarrow A = -7/4,\ B = 15/4$

Answer to question 2:

$A = -7/4$ $\qquad B = 15/4$

3. A coin has been weighted so that heads (H) comes up with a probability of 6/10 and tails (T) comes up with a probability of 4/10. If this coin is tossed 8 times, what is the probability that heads will come up on 4 of the 8 tosses?

Solution: $P(\text{exactly 4 heads}) = {}_8C_4 \cdot (.6)^4(.4)^4 \approx 0.232$

Answer to question 3: 0.232 or around 23.2% of the tosses will have exactly 4 heads.

4. Consider $\Delta\mathbf{ABC}$ with the following information: **AB** = 8, **BC** = 13, **AC** = 6, $\angle\mathbf{C} = 25.33°$. Find the possible measure(s) of $\angle\mathbf{A}$.

Solution:

By Law of Sines: $\frac{\sin A}{13} = \frac{\sin 25.33}{8} \Rightarrow \sin A \approx .695 \Rightarrow A \approx 44.05$ or $A \approx 135.95$. A must be obtuse, so $A \approx 135.95^\circ$.

By Law of Cosines: $13^2 = 6^2 + 8^2 - 2(6)(8)\cos(A) \Rightarrow \cos(A) \approx -.719 \quad A \approx 135.95^\circ$

Answer to question 4:

$\angle\mathbf{A} = \approx 135.95^\circ$

5. For the function $f(x) = 5 \cdot 2^x + 1$, find the limits $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to 1+} f^{-1}(x)$.

Solution:

$f\left(f^{-1}(x)\right) = x \Rightarrow 5 \cdot 2^{f^{-1}(x)} + 1 = x \Rightarrow 2^{f^{-1}(x)} = \frac{x-1}{5} \Rightarrow f^{-1}(x) = \log_2\left(\frac{x-1}{5}\right)$, which has a vertical asymptote at $x = 1$.

Answer to question 5:

$\lim_{x\to-\infty} f(x) = 1$ $\qquad$ $\lim_{x\to 1+} f^{-1}(x) = -\infty$

6. Given vectors **v** and **w** such that $\mathbf{v} \perp \mathbf{w}$ and $|\mathbf{w}| = 1$, prove that $(\mathbf{v} + \mathbf{w}) \cdot \mathbf{w} = 1$. (You may assume any established facts or properties about the dot product.)

Answer to question 6:

Solution:

$(\mathbf{v} + \mathbf{w}) \cdot \mathbf{w} = \mathbf{v} \cdot \mathbf{w} + \mathbf{w} \cdot \mathbf{w}$
$= 0 + \mathbf{w} \cdot \mathbf{w}$ $\qquad$ ($\mathbf{v} \cdot \mathbf{w} = 0$ because $\mathbf{v} \perp \mathbf{w}$)
$= 0 + |\mathbf{w}|^2$ $\qquad$ ($\mathbf{w} \cdot \mathbf{w} = |\mathbf{w}|^2$ for any vector **w**)
$= 0 + 1^2$
$= 1.$

Part B. Medium Problems

8 problems, 4 points each, 32 points total

7. Consider the sequence $\left\{3, -\frac{3}{4}, \frac{3}{16}, -\frac{3}{64}, \cdots\right\}$ and let t_k stand for the kth term of the sequence.

a. Write a recursive definition for the sequence for t_k.

Solution: The sequence is geometric with $r = -\frac{1}{4}$, so $\begin{cases} t_1 = 3 \\ t_k = -\frac{1}{4} t_{k-1} \text{ for } k \geq 2. \end{cases}$

b. Evaluate: $\lim_{n \to \infty} \sum_{j=1}^{n} t_j$.

Solution: The sum of this infinite geometric series is $\frac{3}{1-(-\frac{1}{4})} = \frac{12}{5}$

8. For all parts of this problem, express each of the following trig expressions in terms of some combination of the sine(s) of angle(s) between 0° and 90°. Then evaluate your expression using only the values from the table below.

example: $\sin(-18^\circ)$ *solution*: $\sin(-18^\circ) = -\sin(18^\circ) \approx -(.309) \approx -.309$

A	sinA	A	sinA	A	sinA
1	0.018	31	0.515	61	0.875
2	0.035	32	0.530	62	0.883
3	0.052	33	0.545	63	0.891
4	0.070	34	0.559	64	0.899
5	0.087	35	0.574	65	0.906
6	0.105	36	0.588	66	0.914
7	0.122	37	0.602	67	0.921
8	0.139	38	0.616	68	0.927
9	0.156	39	0.629	69	0.934
10	0.174	40	0.643	70	0.940
11	0.191	41	0.656	71	0.946
12	0.208	42	0.669	72	0.951
13	0.225	43	0.682	73	0.956
14	0.242	44	0.695	74	0.961
15	0.259	45	0.707	75	0.966
16	0.276	46	0.719	76	0.970
17	0.292	47	0.731	77	0.974
18	0.309	48	0.743	78	0.978
19	0.326	49	0.755	79	0.982
20	0.342	50	0.766	80	0.985
21	0.358	51	0.777	81	0.988
22	0.375	52	0.788	82	0.990
23	0.391	53	0.799	83	0.993
24	0.407	54	0.809	84	0.995
25	0.423	55	0.819	85	0.996
26	0.438	56	0.829	86	0.998
27	0.454	57	0.839	87	0.999
28	0.470	58	0.848	88	0.999
29	0.485	59	0.857	89	1.000
30	0.500	60	0.866	90	1.000

a. $\sin(232^\circ)$

Solution:

$\sin(232^\circ) = -\sin(52^\circ) \approx -(.788) \approx -.788$

b. $\cos(82^\circ)$

Solution:

$\cos(82^\circ) = \sin(8^\circ) \approx .139$

c. $\tan(115^\circ)$

Solution:

$$\tan(115^\circ) = \frac{\sin(115^\circ)}{\cos(115^\circ)} = \frac{-\sin(65^\circ)}{\cos(65^\circ)} = \frac{-\sin(65^\circ)}{\sin(25^\circ)} \approx \frac{-.906}{.423} \approx -2.142$$

d. all θ between 0° and 360° such that $\sin(\theta) \approx -.985$

Solution:

sine is (–) in the 3rd and 4th quadrants.
$\sin^{-1}(.985) = 80^\circ$, which is the 1st quadrant reference angle. That puts the 3rd and 4th quadrant angles at 260° and 280° respectively.

9. **a.** Let $L = \log_b 3$, $M = \log_b 4$, and $N = \log_b 5$. Express $\log_b 30$ in terms of some combination of *L*, *M*, and *N*.

Solution: $M = \log_b 4 \Rightarrow M = 2\log_b 2 \Rightarrow \frac{1}{2}M = \log_b 2$.

$\log_b 30 = \log_b(3 \cdot 2 \cdot 5) = \log_b 3 + \log_b 2 + \log_b 5 = L + \frac{1}{2}M + N$

b. Prove the identity $\log_b uv = \log_b u + \log_b v$. You may only assume the definition of logarithm (e.g. $\log_b x = y \Leftrightarrow b^y = x$) in your proof.

(*Hint to get started: Let* $p = \log_b u$ *and* $q = \log_b v$)

Solution: $p = \log_b u \Rightarrow b^p = u$ and $q = \log_b v \Rightarrow b^q = v$.

Therefore, $b^p \cdot b^q = uv \Rightarrow b^{p+q} = uv \Rightarrow \log_b uv = p + q = \log_b u + \log_b v$

10. **a.** What transformations are needed to transform the graph of $f(x) = (27)^x$ into the graph of $g(x) = (\frac{1}{9})^x$?

Solution:

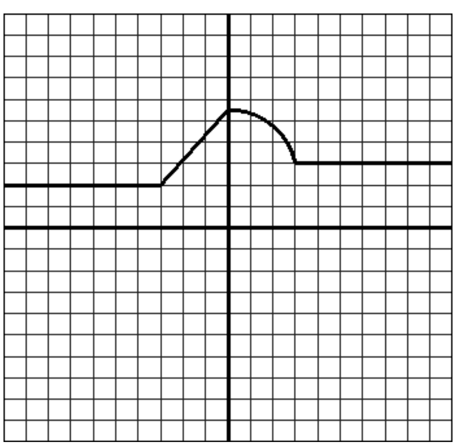
$g(x) = (\frac{1}{9})^x = (3^{-2})^x = 3^{-2x} = (27^{\frac{1}{3}})^{-2x} = 27^{-\frac{2}{3}x} = f(-\frac{2}{3}x)$, so the transformation is a horizontal stretch by a factor of $\frac{3}{2}$ and a flip over the *y*-axis (or vice-versa).

b. Given functions *h*(*x*) and *j*(*x*) with graphs as shown below, identify the transformations that are needed to transform the graph of *h*(*x*) into the graph of *j*(*x*), then write an equation relating function *h* to function *j*.

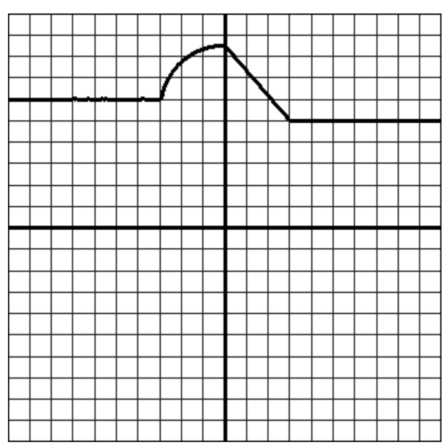
Solution:

To go from *h*(*x*) to *j*(*x*): reflect across the *y*-axis and translate up by 3 units (in either order).

$j(x) = h(-x) + 3$.



graph of *h*(*x*)



graph of *j*(*x*)

11. In this problem about a roller coaster, all numbers are measurements in feet, and all coordinates are in the plane of the roller coaster.

A roller coaster has a set of four consecutive hills that can be modeled by four cycles of a sinusoidal wave. The ride begins at (0, 20), and the peak of the first hill is at (200, 160). The heights of 20 feet and 160 feet are the least and greatest heights on the coaster.

a. Write a sinusoidal function formula that models the hills of the roller coaster.

Solution: period = 400; minimum = 20, maximum = 160; *x*-intercept at a minimum

$y = -70\cos(\frac{2\pi}{400}x) + 90$.

b. It takes a total of 80 seconds to ride the four hills of the roller coaster. Write a pair of parametric equations describing a possible ride on the roller coaster, as a function of time *t* in seconds.

Solution:

It's easiest to choose a linear relationship between *t* and *x*(*t*).
It takes 80 seconds to travel 1600 feet in the *x*-direction, so 20 feet per second.

$$x(t) = 20t$$

$$y(t) = -70\cos(\frac{2\pi}{400}(20t)) + 90$$

12. In the polar coordinate system, consider the points with polar coordinates (r, α) and (s, β). Find a formula for the distance between these two points, in terms of r, s, α, and β. In other words, derive a *polar distance formula*.

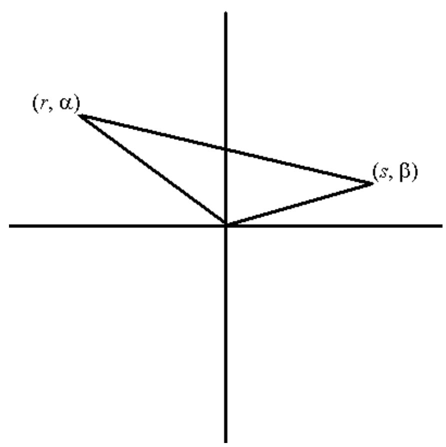
Hint: Use the triangle whose vertices are these two points along with the origin. Start by finding the angle measure at the origin.

Solution:

The angle of the vertex at the origin can be expressed as $\alpha-\beta$. Let d be the distance between the two points. By the Law of Cosines:

$$d^2 = r^2 + s^2 - 2rs\cos(\alpha - \beta)\text{, so}$$

$$d = \sqrt{r^2 + s^2 - 2rs\cos(\alpha - \beta)}$$



13. A bag contains 7 green tennis balls and 3 orange tennis balls.

a. You randomly draw 2 tennis balls out of the bag at the same time. What is the probability that they have different colors?

Solution:

$$p(\text{different colors}) = p(\text{green then orange}) + p(\text{orange then green}) = \left(\tfrac{7}{10}\right)\left(\tfrac{3}{9}\right) + \left(\tfrac{3}{10}\right)\left(\tfrac{7}{9}\right) = \tfrac{7}{15}$$

b. Suppose you play a game of chance involving drawing 2 tennis balls from this bag. Here are the possible payoffs for the game.

colors of tennis balls	payoff
both green	lose 5 cents
different colors	win 1 cent
both orange	win 10 cents

What is your expected amount of winning or losing for a single play of this game?

Solution:

$$p(\text{green and green}) = \left(\tfrac{7}{10}\right)\left(\tfrac{6}{9}\right) = \tfrac{7}{15},\quad p(\text{orange and orange}) = \left(\tfrac{3}{10}\right)\left(\tfrac{2}{9}\right) = \tfrac{1}{15}$$

$$\text{Expected Value} = -5\left(\tfrac{7}{15}\right) + 1\left(\tfrac{7}{15}\right) + 10\left(\tfrac{1}{15}\right) = -\tfrac{6}{5} = -1.2 \text{ cents}$$

14. The second-degree equation $16x^2 - 9y^2 + 32x + 54y - 209 = 0$ represents a non-degenerate hyperbola.

a. Find the equation of the conic in standard form.

Solution:

Completing the square in both *x* and *y* yields:

$$\frac{(x+1)^2}{9} - \frac{(y-3)^2}{16} = 1$$

b. Find the equations of the asymptotes.

Solution:

Both asymptotes pass through the center $(-1,3)$. One has slope of 4/3 and the other has a slope of –4/3. Equations then should be :

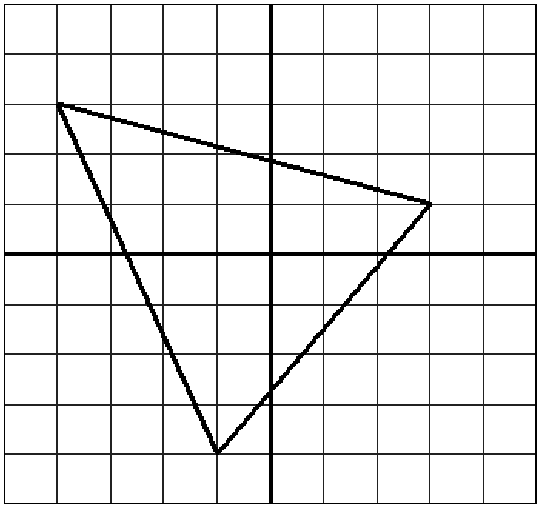
$y - 3 = (4/3)(x + 1)$ and $y - 3 = -(4/3)(x + 1)$

Part C. Long Problems

3 problems, 8 points each, 24 points total

15. The vertices of **ΔABC** are given by the points **A** (3,1), **B** (–1, –4) and **C** (–4,3).

Draw **ΔABC** on the coordinates at right.



a. Find the length of sides **CA** and **CB**.

Solution:

$$\mathbf{CA} = \sqrt{7^2 + 2^2} = \sqrt{53}$$

$$\mathbf{CB} = \sqrt{3^2 + 7^2} = \sqrt{58}$$

b. Find the measure of ∠**C**.

Solution:

$$\cos\angle\mathbf{C} = \frac{\vec{\mathbf{CA}}\cdot\vec{\mathbf{CB}}}{|\vec{\mathbf{CA}}||\vec{\mathbf{CB}}|} = \frac{<7,-2>\cdot<3,-7>}{\sqrt{53}\cdot\sqrt{58}}$$

$$= \frac{7\cdot 3 + (-2)(-7)}{\sqrt{53}\cdot\sqrt{58}} = \frac{35}{\sqrt{53}\cdot\sqrt{58}} \approx 0.6313,$$

so $\angle\mathbf{C} \approx \cos^{-1}(0.6313) \approx 0.888$ or $50.86°$.

Let **D** be a point on vector segment **CB** that is 4/5 of the way from **C** to **B**.

c. Find the coordinates of point **D**.

Solution:

$$\mathbf{D} = \mathbf{C} + \tfrac{4}{5}\left(\vec{\mathbf{CB}}\right) = (-4, 3) + \tfrac{4}{5}<3, -7> = (-8/5, -13/5)$$

d. Δ*ABC* is rotated counterclockwise by 45 degrees to created a new triangle, **Δ A' B' C'**. Find the image of each of the vertices under this transformation:

A (3,1) → **A'** (1.414, 2.828)

B (–1, –4) → **B'** (2.121, –3.536)

C (–4,3) → **C'** (–4.950, –0.707)

Solution:

$$\text{because} \begin{bmatrix} 3 & 1 \\ -1 & -4 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} \cos 45° & \sin 45° \\ -\sin 45° & \cos 45° \end{bmatrix} \approx \begin{bmatrix} 1.414 & 2.828 \\ 2.121 & -3.536 \\ -4.950 & -0.707 \end{bmatrix}$$

(or you could have done three separate multiplications, each 1x2 times 2x2).

16. Let $P(x) = x^4 - 5x^3 + 8x - 40$.

a. Evaluate $P(5)$.

Solution:

$$P(5) = 0$$

b. Factor $P(x)$ as a product of a linear function and a cubic function.

Solution:

$P(x)$ factors, so:

$$P(x) = x^4 - 5x^3 + 8x - 40 = x^3(x-5) + 8(x-5) = (x^3+8)(x-5)$$

OR: Long Division by the known factor from part a. $(x-5)\overline{)x^4 - 5x^3 + 8x - 40}$

OR: Synthetic Division since the divisor is linear.

c. In the complex number system, find all the zeroes of $P(x)$.
You may give your answers in either rectangular or polar form.

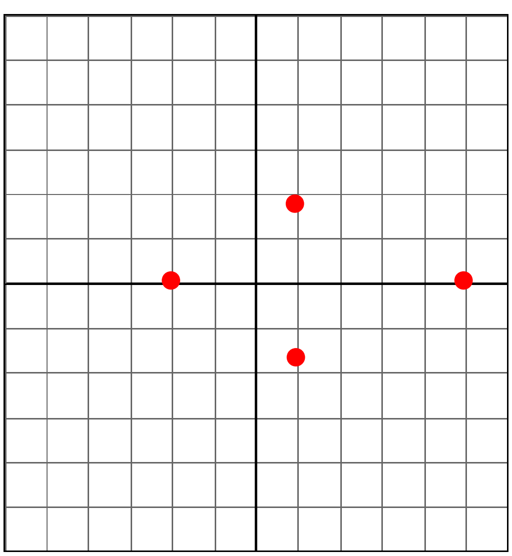
Solution:

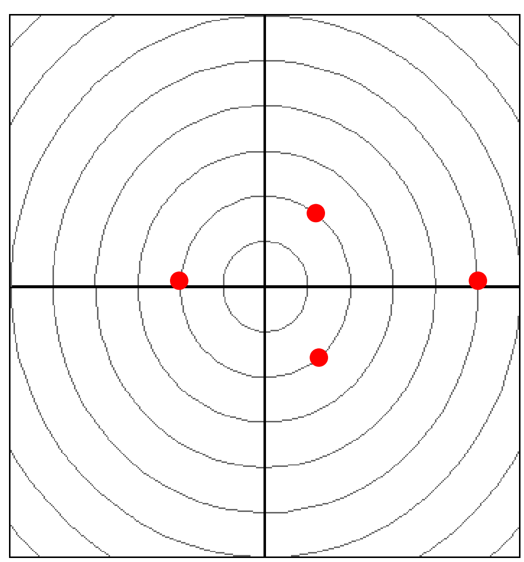
$$(x-5)(x^3+8) = 0$$
$$\Rightarrow (x-5)(x+2)(x^2-2x+4) = 0$$
$$\Rightarrow x = -2 \text{ or } x = 5 \text{ or } x^2 - 2x + 4 = 0$$
$$\Rightarrow x = 5 \text{ or } x = -2 \text{ or } x = 1 \pm i\sqrt{3}$$

OR

$$(x-5)(x^3+8) = 0$$
$$\Rightarrow x = 5 \text{ or } x = 2cis\pi = -2$$
$$\text{or } x = 2cis\left(\frac{\pi}{3}\right)$$
$$\text{or } x = 2cis\left(\frac{5\pi}{3}\right)$$

d. Plot the zeros of P in the complex plane. You may use either the complex rectangular (left) or polar (right) grids below.





17. Fill in the 2nd and 3rd column of the table below.

Function	**Domain/Range Analysis** Determine the *domain* and *range* for each function.	**Even/Odd Analysis** Determine whether each function is *even*, *odd*, or *neither*. Algebraically prove your result
$f(x) = \log_2\lvert x\rvert$	*Domain*: $(-\infty,0)\cup(0,\infty)$ *Range*: $(-\infty,\infty)$	*Circle*: **Even** / Odd / Neither *Proof*: $f(-x) = \log_2\lvert -x\rvert = \log_2\lvert x\rvert = f(x)$ since $f(-x) = f(x)$, f must be even.
$f(x) = \frac{\sin x}{x}$	*Domain*: $(-\infty,0)\cup(0,\infty)$ *Range*: $[-.217,1)$	*Circle*: **Even** / Odd / Neither *Proof*: $f(-x) = \frac{\sin(-x)}{-x} = \frac{-\sin(x)}{-x} = \frac{\sin(x)}{x} = f(x)$ since $f(-x) = f(x)$, f must be even.
$f(x) = \frac{x^3 - 2x^2 - 5x + 6}{x - 1}$	*Domain*: $(-\infty,1)\cup(1,\infty)$ *Range*: $[-25/4,\infty)$	*Circle*: Even / Odd / **Neither** *Proof*: $f(-x) = \frac{-x^3 - 2x^2 + 5x + 6}{-x - 1}$ $-f(x) = \frac{-x^3 + 2x^2 + 5x - 6}{x - 1}$ $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$, so f is neither even nor odd. **OR** Counterexamples $f(2) = -4$ $f(-2) = 0$ Since $f(2) \neq f(-2)$, f is not even. Since $-f(2) \neq f(-2)$, f is not odd.