Name ____

Teacher (circle): Haupt Kelly Kresser Tall Verner

Class block (circle): A B C D E H

Honors Advanced Math
Final Exam 2006 - Solutions

Lexington High School
Mathematics Department

This is a 90-minute exam, but you will be allowed to work for up to 120 minutes.

The exam has 3 parts. Directions for each part appear below.

In total, there are 76 points that you can earn. A letter grade scale will be set by the course faculty after the tests have been graded.

Part A. Short Problems

10 questions, 2 points each, 20 points total

You must write your answers in the answer boxes.

If your answer is correct, you will receive full credit. Showing work is not required.

If your answer is incorrect, you may receive half credit if you have shown some correct work.

A good pace on this part would be to spend around 2-3 minutes per problem.

Part B. Medium Problems

8 problems, 4 points each, 32 points total

Write a complete, clearly explained solution to each problem. Partial credit will be given.

A good pace on this part would be to spend around 4-5 minutes per problem.

Part C. Long Problems

3 problems, 8 points each, 24 points total

Write a complete, clearly explained solution to each problem. Partial credit will be given.

A good pace on this part would be to spend around 8-10 minutes per problem.

Part A. Short Problems

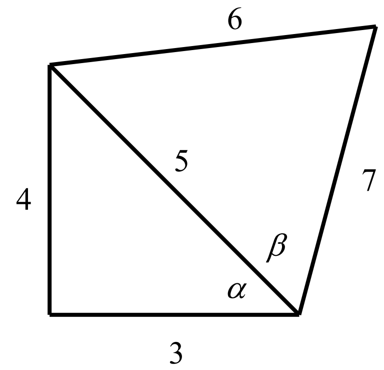
10 problems, 2 points each, 20 points total

1. Find α and β. Angles should be measured to the nearest 0.01 radian.

Note: Diagram not drawn to scale.

$\alpha = \sin^{-1}(4/5) \approx .93$

$$\beta = \cos^{-1}\left(\frac{6^2 - 5^2 - 7^2}{-2 \cdot 5 \cdot 7}\right) \approx 1.00$$



Answer to question 1:

$\alpha \approx .93$ $\beta \approx 1.00$

2. Find each of these limits: $\lim_{x\to\infty} \tan^{-1} x$ and $\lim_{x\to 2+} \ln(x-2)$. Numerical values must be exact.

Both limits can be found graphically.

Answer to question 2:

$\lim_{x\to\infty} \tan^{-1} x = \pi/2$ $\lim_{x\to 2+} \ln(x-2) = -\infty$

3. Find a pair of real numbers r and s such that $2\ln(5) = \dfrac{\log_4 r}{\log_4 s}$.

Left side is: $2\ln(5) = \ln 5^2 = \ln 25 = \dfrac{\ln 25}{\ln e}$

Right side is: $\dfrac{\log_4 r}{\log_4 s} = \dfrac{\ln r/\ln 4}{\ln s/\ln 4} = \dfrac{\ln r}{\ln s}$.

So $r = 25$ and $s = e$ is a possible answer. There are other answers also.

Answer to question 3:

$r = 25$ $s = e$

4. Solving a system of four linear equations in four variables (w, x, y, z) using the augmented matrix method, you obtain the following reduced matrix:

$$\left[\begin{array}{cccc|c} 1 & 0 & -4 & 0 & 5 \\ 0 & 1 & 3 & 0 & -2 \\ 0 & 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

There are infinitely many solutions to this system. Describe the solutions.

Answer to question 4:

$(5+4y, -2-3y, y, 8)$ for any real number y.

5. For a certain type of flower seed, each seed has a 30% chance of growing into a flower. How many seeds would need to be planted to have at least a 90% chance of growing at least one flower?

This will occur when $1-(.70)^n \geq .90$. Solve algebraically or graphically to find $n \geq 7$.

Answer to question 5:

Plant 6 or more seeds to ensure a 90% chance of at least one flower.

6. Let point P have coordinates $(3,-2,-1)$ and plane M have equation $x+2y+2z=15$. Suppose line L passes through point P and is perpendicular to plane M. Write a vector equation for line L.

A vector perpendicular to plane M is $\langle 1,2,2 \rangle$. To pass through $(3,-2,-1)$ and be perpendicular to M, find all points (x,y,z) such that $(x, y, z) = (3,-2,-1) + t\langle 1,2,2 \rangle$.

Answer to question 6:

$(x, y, z) = (3,-2,-1) + t\langle 1,2,2 \rangle$, where $-\infty < t < \infty$

7. Perform the following complex number arithmetic. Note that cis(θ) = cos(θ) + *i* sin(θ).

a. $2\,\text{cis}(\frac{\pi}{2}) + \sqrt{2}\,\text{cis}(\frac{5\pi}{4})$. Express your answer in $a+bi$ form.

$$2\text{cis}(\tfrac{\pi}{2}) + \sqrt{2}\text{cis}(\tfrac{5\pi}{4}) = 2\left(\cos(\tfrac{\pi}{2}) + i\sin(\tfrac{\pi}{2})\right) + \sqrt{2}\left(\cos(\tfrac{5\pi}{4}) + i\sin(\tfrac{5\pi}{4})\right)$$

$$= 2i - 1 - i = -1 + i$$

Answer to question 7a: $-1+i$

b. $\left(\sqrt{3}\,\text{cis}\left(\frac{\pi}{3}\right)\right)^{12}$ Express your answer in $a+bi$ form.

$$\left(\sqrt{3}^{\,12}\text{cis}\left(\tfrac{12\pi}{3}\right)\right) = 3^6\,\text{cis}(4\pi) = 729$$

Answer to question 7b: 729 + 0*i*

8. Write a Cartesian (rectangular) equation for the ellipse shown below. The two points with coordinates labeled are the foci (focal points) of the ellipse.

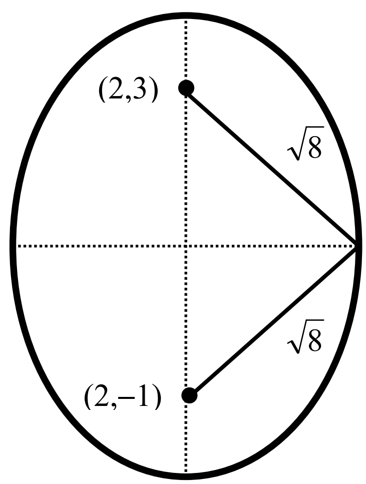
Center: (2,1)

Focal Length: 2

Semi-major axis: $\sqrt{8}$

Semi-minor axis: $\sqrt{8-4} = 2$

$$\frac{(x-2)^2}{4} + \frac{(y-1)^2}{8} = 1$$



Answer to question 8:

$$\frac{(x-2)^2}{4} + \frac{(y-1)^2}{8} = 1$$

9. Three one-to-one functions, F, G, and H, are defined as follows:

F: $F(x) = \dfrac{1}{x-2}$

G: $G(x) = 1 - \sqrt{x}$

H:

x	1	2	3	4	5
$H(x)$	-1	9	7	2	-2

Evaluate each of the following expressions.

- $(F \circ H^{-1})(2)$
- $(F^{-1} \circ G^{-1})(-3)$

Answer to question 9:

$(F \circ H^{-1})(2) = \dfrac{1}{2}$ $(F^{-1} \circ G^{-1})(-3) = \dfrac{33}{16}$

10. At right is the graph of a sinusoidal function, $y = f(x)$.

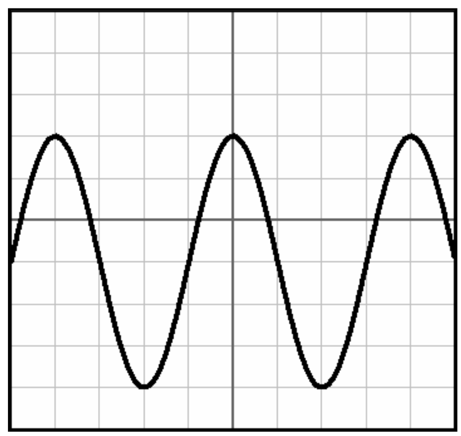
Write an equation for $f(x)$ involving **sine**.

Amplitude: 3

Wave axis: -1

Period: $4 = \dfrac{2\pi}{\text{"B"}} \Rightarrow B = \dfrac{\pi}{2}$

$f(x) = 3\sin\left(\frac{\pi}{2}(x+1)\right) - 1$



Answer to question 10:

$$f(x) = 3\sin\left(\tfrac{\pi}{2}(x+1)\right) - 1$$

Part B. Medium Problems

8 problems, 4 points each, 32 points total

11. A cube is labeled with a colored number on each face. Specifically, this is how the 6 faces are labeled:

green 1, blue 2, red 3, red 4, purple 5, green 6.

a. Suppose that you rolled the cube twice. Find the probability that you would get the same color both times.

$P(\text{ Both the same color}) = P(BB) + P(GG) + P(RR) + P(PP)$

$= 1/36 + 4/36 + 4/36 + 1/36 = 10/36$

b. Suppose that you rolled the cube 10 times.

Let $P(x)$ = the probability of rolling the "purple 5" x-out-of-10 times. Evaluate $P(2)$.

$P(x) = {}_{10}C_x \cdot (1/6)^x \cdot (5/6)^{10-x}$

$P(2) = {}_{10}C_2 \cdot (1/6)^2 \cdot (5/6)^8 \approx .291$

12. The Earth's atmospheric pressure is related to altitude. The pressure at the Earth's surface (sea level) is about 14.7 pounds per square inch (psi) and the pressure at 2000 feet above sea level is approximately 13.5 psi.

a. Find an expression in the form of $P(a) = P_0 e^{k \cdot a}$ that models the atmospheric pressure, $P(a)$, as an exponential function of the altitude, a, above sea level.

$P_0 = 14.7$

Find k: $P(2000) = 14.7 \cdot e^{k \cdot 2000} = 13.5 \Rightarrow k \approx -.0000426$

$P(a) = 14.7 \cdot e^{-.0000426 \cdot a}$

b. Long ago, the Mediterranean Sea was dry, leaving a "valley" 10,000 feet below sea level. What would the air pressure have been at the bottom of this valley?

$P(-10{,}000) = 14.7 \cdot e^{-.0000426 \cdot -10{,}000} \approx 22.5\, psi$

13. Consider the conic section, *C*, defined parametrically by the equations $C:\begin{cases} x = -1+\sec t \\ y = 2+3\tan t \end{cases}$.

a. Find a Cartesian (rectangular) equation for *C*.

Center: (-1,2)

Semi-Transverse Axis: 1

Semi-Conjugate Axis: 3 $\Rightarrow \dfrac{(x+1)^2}{1} - \dfrac{(y+1)^2}{9} = 1$

b. Find the coordinates for the foci (focal points) of *C*.

$(\text{Focal Length})^2 = 1^2 + 3^2 \quad \Rightarrow \text{Focal Length} = \sqrt{10}$

Coordinates of foci: $(-1-\sqrt{10}\,, 2)$ and $(-1+\sqrt{10}\,, 2)$

14. Consider the complex number $z = \cos\theta + i\sin\theta = \text{cis}\,\theta$.

a. Expand z^4 with the Binomial Theorem.

$(\cos\theta + i\sin\theta)^4 = \cos^4\theta + 4\cos^3\theta\cdot i\sin\theta + 6\cos^2\theta\cdot i^2\sin^2\theta + 4\cos\theta\cdot i^3\sin^3\theta + i^4\sin^4\theta$

$= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta + 4\cos^3\theta\cdot i\sin\theta + +4\cos\theta\cdot i^3\sin^3\theta$

$= \left(\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta\right) + i\left(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta\right)$

b. Find the power z^4 with DeMoivre's Theorem (polar power formula).

$(1\text{cis}\,\theta)^4 = 1^4\,\text{cis}(4\theta) = \text{cis}(4\theta) = \cos(4\theta) + i\sin(4\theta)$

c. Use the results of each of these calculations to write a trigonometric identity for $\cos(4\theta)$, in terms of powers of $\cos\theta$ and $\sin\theta$.

Since parts a and b both yield expressions for z^4 in a + bi form, equate the real part of each to obtain the desired identity:

$\cos(4\theta) = \left(\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta\right)$

15. Consider two sequences, a_n and b_n, defined as follows:

$$a_n = \begin{cases} 2 \cdot a_{n-1} & \text{for } n \ge 2 \\ 1 & \text{for } n = 1 \end{cases} \qquad \text{and} \qquad b_n = \log_3(a_n).$$

Find exact values for $\sum_{n=1}^{25} a_n$ and $\sum_{n=1}^{25} b_n$.

$$\sum_{n=1}^{25} a_n = 1 + 2 + 4 + \cdots + 2^{24} = \text{finite geometric series} = \frac{1(1-2^{25})}{1-2} = 33{,}554{,}431.$$

$$\sum_{n=1}^{25} b_n = \log_3 1 + \log_3 2 + \log_3 2^2 + \log_3 2^3 + \cdots + \log_3 2^{24} = \text{finite arithmetic series}$$

$$\Rightarrow \sum_{n=1}^{25} b_n = 0 + 1 \cdot \log_3 2 + 2 \cdot \log_3 2 + 3 \cdot \log_3 2 + \cdots + 24 \cdot \log_3 2$$

$$\Rightarrow \sum_{n=1}^{25} b_n = \frac{25(0 + 24\log_3 2)}{2} = 300\log_3 2 \approx 189.28$$

16. Consider this transformation: a 1 radian clockwise rotation around the origin. Answer these questions using matrix methods. Show your work.

a. Find the image of the point (2, –1).

$$\begin{bmatrix} \cos 1 & \sin 1 \\ -\sin 1 & \cos 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \end{bmatrix} \approx \begin{bmatrix} .239 \\ -2.223 \end{bmatrix}$$

b. Find the point (a, b) whose image is (3, 4).

Either rotate (3,4) 1 radian counter-clockwise:

$$\begin{bmatrix} \cos 1 & -\sin 1 \\ \sin 1 & \cos 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \end{bmatrix} \approx \begin{bmatrix} -1.745 \\ 4.685 \end{bmatrix}$$

Or solve this matrix equation:

$$\begin{bmatrix} \cos 1 & \sin 1 \\ -\sin 1 & \cos 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} \approx \begin{bmatrix} 3 \\ 4 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} \approx \begin{bmatrix} -1.745 \\ 4.685 \end{bmatrix}.$$

17. One of these equations is a trigonometric identity:

$$\frac{\cos x}{1+\sin x} = \frac{1-\cos x}{\sin x} \quad \text{or} \quad \frac{\sin x}{1-\cos x} = \frac{1+\cos x}{\sin x}$$

Circle the identity and prove it using a sequence of equal expressions that starts with one side of the equation and ends with the other.

$$\frac{\sin x}{1-\cos x} = \frac{\sin x(1+\cos x)}{1-\cos^2 x} = \frac{\sin x(1+\cos x)}{\sin^2 x} = \frac{(1+\cos x)}{\sin x}.$$

18. Write a possible function formula for a function $f(x)$ satisfying all of the following conditions:

- $f(x)$ is a polynomial with integer coefficients.
- $f(x) \geq 0$ for all $x \geq 1$.
- $f(1) = 2$.
- $f(2) = 0$.
- $f(4 - i) = 0$.
- $(x + 6)$ is a factor of $f(x)$.
- $\lim_{x\to-\infty} f(x) = -\infty$.

A polynomial with the right zeros and end behavior:

$$f(x) = (x-2)^2(x+6)(x-(4-i))(x-(4+i)) = (x-2)^2(x+6)(x^2-8x+17)$$

Scaling f so that it passes through (1,2) gives:

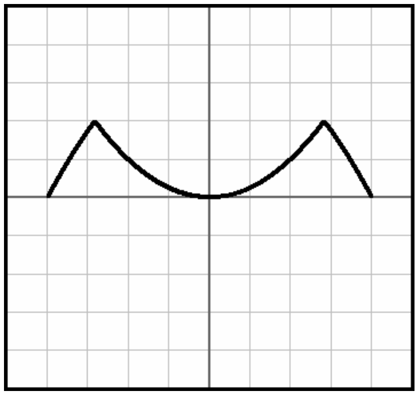
$$f(x) = \frac{1}{35}(x-2)^2(x+6)(x^2-8x+17)$$

Part C. Long Problems

3 problems, 8 points each, 24 points total

19. Consider the function f defined by the graph at right.

The graph of $y = f(x)$



window: [-5,5] by [-5,5]

a. Let $h(x) = -f(x-1)+2$. Find the domain and range of h.

Domain: $[-3,5]$

Range: $[0,2]$

b. On the grid at right, sketch the graph of the function $y = f(x)+x$

Use this grid for part **b.**

c. List a sequence of transformations that would turn the graph of $y = f(x)$ into the graph of $y = f(2x+1)$.

1. Horizontal Shrink by 1/2 (1/2 as wide).
2. Shift left by 1/2 .

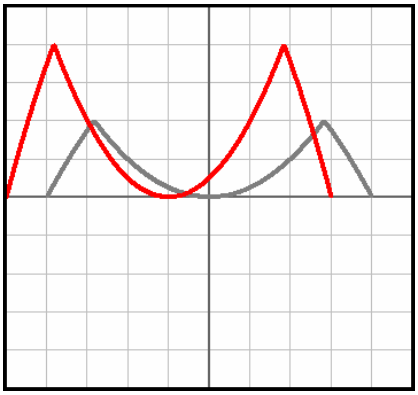
or equivalently

1. Shift left by 1.
2. Horizontal Shrink by 1/2 (1/2 as wide).

d. Let transformation $T = \begin{cases} x' = x-1 \\ y' = 2y \end{cases}$.

On the grid at right, sketch the graph of the transformation of $f(x)$ by T.

Use this grid for part **d.**



20. For both parts of this problem, solve the given equation using the standard analytic techniques appropriate for each type of problem, *i.e., for trigonometric equations use identities, for polynomial equations factor completely.*

<u>No credit will be given for graphical or calculator approximations on this page.</u>

a. Find all θ on $[0, 2\pi]$ such that $\sin(2\theta) + \sin(4\theta) = 0$.

$\sin(2\theta) + 2\sin(2\theta)\cos(2\theta) = 0$

$\sin(2\theta)[1 + 2\cos(2\theta) = 0$

$\sin(2\theta) = 0$ *or* $\cos(2\theta) = -1/2$

$2\theta = 0 + \pi \cdot n$ *or* $2\theta = \left\{\frac{2\pi}{3}, \frac{4\pi}{3}\right\} + 2\pi \cdot n$, where n is a positive integer.

$\theta = 0 + \frac{\pi}{2} \cdot n$ *or* $\theta = \left\{\frac{1\pi}{6}, \frac{2\pi}{6}\right\} + \pi \cdot n$.

Within the specified domain: $\theta = \left\{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\right\}$ or $\left\{\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}\right\}$.

b. Find all x such that $x^3 + x - 10 = 0$.

A good guess yields that x = 2 is a zero (confirmed via Remainder Thm, or division).

$(x-2)(x^2 + 2x + 5) = 0$

$x = 2$ or $x = \frac{-2 \pm \sqrt{4-20}}{2}$

The three solutions are $x = 2$, $x = -1 + 2i$, $x = -1 - 2i$.

21. Consider triangle ABC where $A = (2, 13)$, $B = (9, -11)$, and vector $\overrightarrow{AC} = \langle 15,20 \rangle$.

a. Prove that triangle ABC is isosceles (has two equal sides).

$\overrightarrow{AB} = \langle 7,-24 \rangle$, so $\left|\overrightarrow{AB}\right| = 25 = \left|\overrightarrow{AC}\right|$. Two sides of this triangle have the same length. That's isosceles.

b. Find the measure of angle A.

$$\cos A = \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{\left|\overrightarrow{AB}\right|\left|\overrightarrow{AB}\right|} = \frac{7 \cdot 15 - 24 \cdot 20}{25^2} \Rightarrow < \mathrm{A} \approx 2.214$$

c. Find the area of triangle ABC.

$$A = \frac{1}{2} \cdot 25 \cdot 25 \cdot \sin A \approx 250$$

d. Find the coordinates of the point where the medians of triangle ABC intersect.

Hint: The medians of a triangle intersect at a point that divides each median in a 2:1 ratio.

$$(2,13) + \frac{2}{3}\langle 11,-2 \rangle = \left(\frac{28}{3}, \frac{35}{3} \right)$$

e. Draw the altitude from point A to side BC. Find the coordinates of the point where this altitude intersects side BC.

Let AD be the altitude. Methods vary: $D = (13,11)$.