

Name _____

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Honors Advanced Math Final Exam 2008

**Lexington High School
Mathematics Department**

This is a 90-minute exam, but you will be allowed to work for up to 120 minutes.

The exam has 3 parts. Directions for each part appear below.

In total, there are 58 points that you can earn. A letter grade scale will be set by the course faculty after the tests have been graded.

Part A. Short Problems

7 questions, 2 points each, 14 points total

You must write your answers in the answer boxes.

If your answer is correct, you will receive full credit. Showing work is not required.

If your answer is incorrect, you may receive half credit if you have shown some correct work.

A good pace on this part would be to spend 2-4 minutes per problem.

Part B. Medium Problems

5 problems, 4 points each, 20 points total

Write a complete, clearly explained solution to each problem. Partial credit will be given.

A good pace on this part would be to spend 4-6 minutes per problem.

Part C. Long Problems

3 problems, 8 points each, 24 points total

Write a complete, clearly explained solution to each problem. Partial credit will be given.

A good pace on this part would be to spend 8-12 minutes per problem.

Part A. Short Problems

7 problems, 2 points each, 14 points total

1. Find the solution to the equation $7 \cos(\theta) + 3 = 1$ in the interval $\pi < \theta < \frac{3\pi}{2}$.

Your answer should be given as an exact expression involving an inverse trigonometric function.

Note that the desired answer is in quadrant 3

General solution:

$$\cos(\theta) = -2/7$$

$$\theta = \pm \text{Cos}^{-1}(-2/7) + 2\pi k$$

Where $\text{Cos}^{-1}(-2/7)$ is the principal solution in the interval $[0, \pi]$.

Since $-2/7$ is negative, the principal solution will be quadrant 2. $-\text{Cos}(-2/7)$ is a negative angle in quadrant 3. That is $-\pi < -\text{Cos}^{-1}(-2/7) < -\pi/2$. So the desired answer is $-\text{Cos}^{-1}(-2/7) + 2\pi$

Answer to question 1:

$$\theta = -\text{Cos}^{-1}(-2/7) + 2\pi$$

2. Find the sum of the finite series $e^{3k} + e^{5k} + e^{7k} + e^{9k} + e^{11k} + \dots + e^{99k}$.

You do not need to simplify your answer.

Formula for a finite geometric series: $\sum_{i=1}^n t_i \cdot r^{(i-1)} = t_1 \cdot \frac{1-r^n}{1-r}$

In this case, $t_1 = e^{3k}$ and $r = e^{2k}$.

To find n , note that $e^{99k} = e^{3k} \cdot (e^{2k})^{(n-1)}$.

So $99k = 3k + 2k(n-1)$ and $n = 49$.

So the sum is $e^{3k} \frac{1-(e^{2k})^{49}}{1-e^{2k}}$

Answer to question 2: The sum of the series is

$$e^{3k} \frac{1-(e^{2k})^{49}}{1-e^{2k}}$$

3. Let $g(x) = \sin x$. The following sequence transforms the graph of $g(x)$ into the graph of $h(x)$.
- First, shift left by 6
 - Second, stretch horizontally by a factor of 2.
 - Third, shift right by 6

Write a function formula for $h(x)$.

- First: $y = \sin(x + 6)$
- Second: $y = \sin\left(\frac{1}{2}x + 6\right)$
- Third: $y = \sin\left(\frac{1}{2}(x - 6) + 6\right) = \sin\left(\frac{1}{2}x + 3\right)$

Answer to question 3:

$$h(x) = \sin\left(\frac{1}{2}x + 3\right)$$

4. The equation $x = 2y^2 - 12y + 19$ describes a parabola. Find the equation of this parabola's directrix line.

Complete the square to get $(x - 1) = 2(y - 3)^2$

This is a parabola with a horizontal axis, opening to the right, with vertex (1, 3)

Since $2 = \frac{1}{4p}$, $p = \frac{1}{8}$. So the directrix is $x = 1 - p = 1 - 1/8 = 7/8$

Answer to question 4: The equation of the directrix line is

$$x = 7/8$$

5. Rational function $\frac{x^4 - 7x^3 + 11x^2 + 5x - 9}{x^2 - 4x - 2}$ has a quadratic function as an asymptote (that is, as an end behavior model). Find this quadratic function.

Do the polynomial division. The quotient is the asymptote. The quotient is:

$$y = x^2 + 3x + 25$$

Answer to question 5: The quadratic function is

$$y = x^2 + 3x + 25$$

6. Your car is standing still on a straight, level stretch of highway. At time $t = 0$ seconds, you push the gas pedal to the floor. After 5 seconds you are going 40 kilometers per hour. Let D be the *difference* between your car's speed and its maximum speed of 120 km/hour. Assume that D decreases exponentially with t . How long will it take you to reach 110 km/hour? (Give your answer as a decimal approximation accurate to the nearest 0.01.)

$$D(t) = D_0 \cdot b^t$$

Where $D(t)$ is the difference at time t seconds

D_0 is the initial difference, 120 km/hr

b is the exponential decay multiplier ($b < 1$)

Since $D(5) = 80 = 120 \cdot b^5$,

$$b = \left(\frac{2}{3}\right)^{1/5} \text{ and } D(t) = 120 \cdot \left(\frac{2}{3}\right)^{t/5}$$

Solving the equation $120 - 110 = 10 = 120 \cdot \left(\frac{2}{3}\right)^{t/5}$

$$\text{gives } t = \frac{5 \log(1/12)}{\log(2/3)} \approx 30.64$$

Answer to question 6: The amount of time to reach 110 km/hour is

30.64 seconds

7. Find the function formula of polynomial $P(x)$ that satisfies the following conditions:

- $P(x)$ has degree 4.
- $P(x)$ is an even function.
- $P(0) = 3$.
- $\lim_{x \rightarrow -\infty} P(x) = \infty$.
- The graph of $P(x)$ has an x -intercept of 2.
- The graph of $P(x)$ is tangent to the x -axis.

Note the following implications of the bullets above:

- The first and fourth mean the leading coefficient is positive
- The second and fifth mean -2 is also an intercept.
- The sixth means at least one of the zeros has even multiplicity
- The second means y -axis symmetry means that if one zero has even multiplicity, its mirror image must also have even multiplicity.
- All of the above mean that 2 and -2 are the only intercepts, and both have multiplicity 2.

So: $P(x) = a(x - 2)^2(x + 2)^2$

Plugging in $P(0) = 3$ and solving for a gives $a = 1/12$.

Answer to question 7:

$$P(x) = \frac{1}{12}(x - 2)^2(x + 2)^2$$

Part B. Medium Problems

5 problems, 4 points each, 20 points total

8. Consider the vectors $\mathbf{v} = \langle 4, 3 \rangle$ and $\mathbf{w} = \langle 2, -6 \rangle$.
- a. Find real numbers j and k such that $j\mathbf{v} + k\mathbf{w} = \langle 2, 14 \rangle$.
(Give your answers as exact values.)

Setting up a system of equations gives:

$$4j + 2k = 2$$

$$3j - 6k = 14$$

Solving this system (substitution, linear combinations, matrix/RREF) gives:

$$j = 4/3, \quad k = -5/3$$

- b. Suppose \mathbf{v} and \mathbf{w} are regarded as one-column matrices, $\mathbf{v} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$.
Find a 2-by-2 matrix M such that $M\mathbf{v} = \mathbf{w}$.

Letting $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we want $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$

As a system, this is two equations in four unknowns. We should be able to pick two unknowns arbitrarily, and solve for the other two. Letting $b = c = 0$ gives:

$$M = \begin{bmatrix} 1/2 & 0 \\ 0 & -2 \end{bmatrix}$$

9. The moon goes through a cycle of phases from full (completely lit) to new (completely unlit) and back to full every 29 days. Consider the portion of the moon that is lit (when viewed from the earth as a circular disc) as a function of time. It is a fact that this function is approximately sinusoidal, so let t represent time in days and let $f(t)$ represent a sinusoidal function that models the portion of the moon that is lit. The range of $f(t)$ is from 0 (unlit) to 1 (fully lit). Further suppose that the moon is fully lit on when $t = 4.00$.
- a. Write a function formula for sinusoidal function $f(t)$.

Amplitude: 0.5

Phase shift: 4 right for cosine

Vertical shift: 0.5

Period: 29 days, $B = 2\pi/29$

$$f(t) = 0.5 \cos\left(\frac{2\pi}{29}(t - 4)\right) + 0.5$$

- b. Using the $f(t)$ model, find the first 3 times (smallest 3 positive values of t) at which $\frac{3}{4}$ of the moon's disc will be lighted. (Give your answer as a decimal approximation accurate to the nearest 0.01.)

Since no solution technique is specified, we can solve graphically.

$$t = 8.83, 28.17, 37.83 \quad (\text{Note that the first and third answers are exactly one period apart.})$$

10. The equations of two planes are given below:

$$\text{Plane } P_1 : 2x + 3y + 5z = -2$$

$$\text{Plane } P_2 : 5x + y - 7z = -18$$

a. Find the vector equation of the line that is formed where the two planes intersect.

We want points (x, y, z) that satisfy both equations. Writing this as a matrix gives:

$$\left[\begin{array}{ccc|c} 2 & 3 & 5 & -2 \\ 5 & 1 & -7 & -18 \end{array} \right]. \text{ Using the calculator's RREF command on this matrix gives:}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & -4 \\ 0 & 1 & 3 & 2 \end{array} \right]$$

This is the reduced matrix for a line.

Let $z = t$, then:

$$x = 2t - 4$$

$$y = -3t + 2$$

The vector equation of the line is:

$$(x, y, z) = (-4, 2, 0) + t\langle 2, -3, 1 \rangle$$

b. Find the acute angle formed by the two planes. (Give your answer as a decimal approximation accurate to the nearest 0.01 degree or 0.01 radian.)

The angle between the planes is congruent to the angle between the vectors perpendicular to each plane. Using the dot product formula for the angle between two vectors gives:

$$\theta = \cos^{-1} \left(\frac{\langle 2, 3, 5 \rangle \cdot \langle 5, 1, -7 \rangle}{\|\langle 2, 3, 5 \rangle\| \cdot \|\langle 5, 1, -7 \rangle\|} \right) \approx 114.34^\circ \approx 2.00 \text{ rad}$$

This angle is obtuse. So the acute angle is $\theta \approx 65.66^\circ \approx 1.15 \text{ rad}$

11. Suppose $z = a + bi$ and $\bar{z} = a - bi$ (the conjugate of z). If $g(z) = \frac{3}{z}$, show that $g(\bar{z}) = \overline{g(z)}$.

We want to show that $\frac{3}{\bar{z}} = \overline{\left(\frac{3}{z}\right)}$. Note that we may not assume $\frac{1}{\bar{z}} = \overline{\left(\frac{1}{z}\right)}$, as this would make the problem trivial.

$$\frac{3}{\bar{z}} = \frac{3}{a - bi} = \frac{3}{a + bi} \cdot \frac{a - bi}{a - bi} = \frac{3a - 3bi}{a^2 + b^2}$$

By a similar calculation: $\frac{3}{z} = \frac{3a + 3bi}{a^2 + b^2}$

Taking the conjugate gives $\overline{\left(\frac{3}{z}\right)} = \frac{3a - 3bi}{a^2 + b^2}$

12. For the following proof, the only trigonometric identities you may assume are:

- $\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$,
- all the Pythagorean identities.

Prove that $\sin^2(x) = \frac{1 - \cos(2x)}{2}$.

$$\begin{aligned} \cos(2x) &= \cos(x + x) \\ &= \cos x \cdot \cos x - \sin x \cdot \sin x \\ &= \cos^2 x - \sin^2 x \\ &= (1 - \sin^2 x) + \sin^2 x \\ &= 1 - 2\sin^2 x \end{aligned}$$

Equating the first and last expressions in this chain and solving for $\sin^2 x$ gives:

$$\sin^2 x = \frac{\cos(2x) - 1}{-2} = \frac{1 - \cos(2x)}{2}$$

Part C. Long Problems

3 problems, 8 points each, 24 points total

13. Consider the linear transformation T of the plane described by this matrix equation:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Let E stand for the ellipse whose axes are the line segments from $(1, 3)$ to $(7, 3)$ and from $(4, -2)$ to $(4, 8)$.

In answers to the following questions, numbers may be given as decimal approximations accurate to the nearest 0.01.

a. Find the input point whose image under transformation T is $(x', y') = (-2, 4)$.

Note that $\begin{bmatrix} x \\ y \end{bmatrix} = T^{-1} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1/3 \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x' \\ y' \end{bmatrix}$

Plugging in $(-2, 4)$ gives $(x, y) = (4/3, 2)$

b. Find the coordinates of the foci (focal points) of ellipse E .

Note that the vertical axis is 10 units long and the horizontal axis is 6. So the major axis is vertical, the center is $(4, 3)$, and the equation of the ellipse is:

$$\frac{(x-4)^2}{9} + \frac{(y-3)^2}{25} = 1$$

Since $a = 5$ and $b = 3$, $c = 4$, and the foci are 4 units above and below the center:

$$F_1 = (4, 7), F_2 = (4, -1)$$

c. Write a pair of parametric equations describing ellipse E .

$$x = 3\cos\theta + 4$$

$$y = 5\sin\theta + 3$$

$$0 \leq \theta < 2\pi$$

d. Suppose that transformation T is applied to ellipse E . The image will be another ellipse. Find a pair of parametric equations describing the image ellipse.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 3 & 0 \end{bmatrix} \cdot \begin{bmatrix} 3\cos\theta + 4 \\ 5\sin\theta + 3 \end{bmatrix} = \begin{bmatrix} -5\sin\theta - 3 \\ 9\cos\theta + 12 \end{bmatrix}$$

$$x' = -5\sin\theta - 3, \quad y' = 9\cos\theta + 12$$

14. In quadrilateral $ABCD$ (shown below), $AB = 6$, $BC = 5$, $CD = 5$, $AD = 9$, and the measure of $\angle DCB$ is twice the measure of $\angle DAB$.

You should not assume that the diagram is drawn exactly to scale, but do assume that the quadrilateral is concave, and that $\angle DCB$ is an obtuse angle.

Find the area of quadrilateral $ABCD$. (Give your answer as a decimal approximation accurate to the nearest 0.01.)

Draw line segment DB . Using the law of cosines twice gives:

$$(DB)^2 = 9^2 + 6^2 - 2(9)(6)\cos x = 5^2 + 5^2 - 2(5)(5)\cos 2x$$

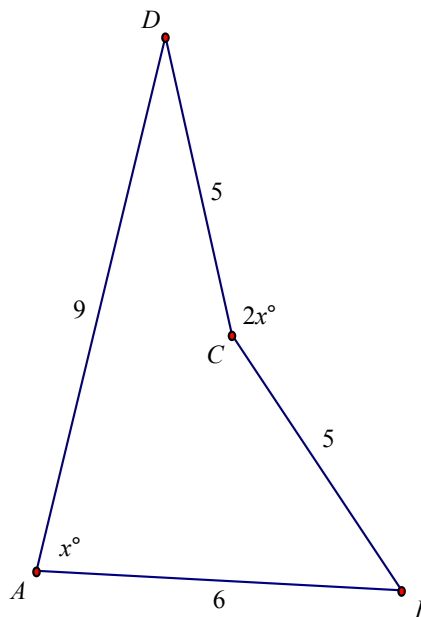
This gives $117 - 108\cos x = 50 - 50\cos 2x$

This equation can be solved algebraically or graphically to give $x = 27.29^\circ$ or $x = 78.98^\circ$

Since $2x$ must be obtuse, x must be the larger of these two values.

$$\begin{aligned} \text{Area } \triangle DAB &= 0.5(9)(6)\sin 78.98^\circ = 26.5 \\ \text{Area } \triangle DCB &= 0.5(5)(5)\sin(2 \cdot 78.98) = 4.69 \end{aligned}$$

The area of the quadrilateral is the difference of these two areas, or 21.81



NOTE: It can be proven that $\triangle DAB$ must be an inscribed triangle in a circle centered at C with radius 5. So segment AC has length 5. We can then find the areas of $\triangle ACB$ and $\triangle ADC$ using Heron's formula. The area of the quadrilateral is the sum of these two.

15. These facts are given about a triangle ABC in the plane:

- The rectangular coordinates of point A are $(x, y) = (1, 2)$.
- Vector $\overrightarrow{AB} = \langle 4, 6 \rangle$.
- Point C is in the 4th quadrant (lower right quadrant).
- Length $AC = 9$.
- Angle $BAC = \frac{2\pi}{5}$ radians.

In answering the following questions, you may give numbers as decimal approximations accurate to the nearest 0.01.

a. Find vector \overrightarrow{AC} in component form.

$$B = A + \overrightarrow{AB} = (5, 8)$$

On graph paper, plot A, B and draw AB

Estimate the location of C and draw AC

Draw a horizontal segment AD to the right of A

Angle BAD is the direction angle of vector AB, or $\angle BAD = \tan^{-1} 4/6 \approx .9828rad$.

$$\angle DAC = 2\pi/5 - \angle BAD \approx .2738rad$$

The direction angle of vector AC is the negative of this value.

$$\overrightarrow{AC} \approx \langle 9 \cos-.2738, 9 \sin-.2738 \rangle \approx \langle 8.664, -2.434 \rangle$$

b. Find the polar coordinates (r, θ) of point C.

The Cartesian coordinates of C are $C = A + \overrightarrow{AC} = (9.664, -0.434)$

$$r = \sqrt{9.664^2 + .434^2} = 9.67$$

$$\theta = \tan^{-1} \frac{-.434}{9.664} = -0.04rad$$