Name ____

Teacher (circle): Haupt Kelly Normile Verner

Class block (circle): A B C E G H

Honors Advanced Math
Final Exam 2008

Lexington High School
Mathematics Department

This is a 90-minute exam, but you will be allowed to work for up to 120 minutes.

The exam has 3 parts. Directions for each part appear below.

In total, there are 58 points that you can earn. A letter grade scale will be set by the course faculty after the tests have been graded.

Part A. Short Problems

7 questions, 2 points each, 14 points total

You must write your answers in the answer boxes.

If your answer is correct, you will receive full credit. Showing work is not required.

If your answer is incorrect, you may receive half credit if you have shown some correct work.

A good pace on this part would be to spend 2-4 minutes per problem.

Part B. Medium Problems

5 problems, 4 points each, 20 points total

Write a complete, clearly explained solution to each problem. Partial credit will be given.

A good pace on this part would be to spend 4-6 minutes per problem.

Part C. Long Problems

3 problems, 8 points each, 24 points total

Write a complete, clearly explained solution to each problem. Partial credit will be given.

A good pace on this part would be to spend 8-12 minutes per problem.

Part A. Short Problems

7 problems, 2 points each, 14 points total

1. Find the solution to the equation $7\cos(\theta) + 3 = 1$ in the interval $\pi < \theta < \frac{3\pi}{2}$.
Your answer should be given as an exact expression involving an inverse trigonometric function.

Answer to question 1:

$\theta =$

2. Find the sum of the finite series $e^{3k} + e^{5k} + e^{7k} + e^{9k} + e^{11k} + \cdots + e^{99k}$.
You do not need to simplify your answer.

Answer to question 2: The sum of the series is

3. Let $g(x) = \sin x$. The following sequence transforms the graph of $g(x)$ into the graph of $h(x)$.

- First, shift left by 6
- Second, stretch horizontally by a factor of 2.
- Third, shift right by 6

Write a function formula for $h(x)$.

Answer to question 3:

$h(x) =$

4. The equation $x = 2y^2 - 12y + 19$ describes a parabola. Find the equation of this parabola's directrix line.

Answer to question 4: The equation of the directrix line is

5. Rational function $\frac{x^4 - 7x^3 + 11x^2 + 5x - 9}{x^2 - 4x - 2}$ has a quadratic function as an asymptote (that is, as an end behavior model). Find this quadratic function.

Answer to question 5: The quadratic function is

6. Your car is standing still on a straight, level stretch of highway. At time $t = 0$ seconds, you push the gas pedal to the floor. After 5 seconds you are going 40 kilometers per hour. Let D be the *difference* between your car's speed and its maximum speed of 120 km/hour. Assume that D decreases exponentially with t. How long will it take you to reach 110 km/hour? (Give your answer as a decimal approximation accurate to the nearest 0.01.)

Answer to question 6: The amount of time to reach 110 km/hour is

7. Find the function formula of polynomial $P(x)$ that satisfies the following conditions:

- $P(x)$ has degree 4.
- $P(x)$ is an even function.
- $P(0) = 3$.
- $\lim_{x \to -\infty} P(x) = \infty$.
- The graph of $P(x)$ has an x-intercept of 2.
- The graph of $P(x)$ is tangent to the x-axis.

Answer to question 7:

$P(x) =$

Part B. Medium Problems

5 problems, 4 points each, 20 points total

8. Consider the vectors **v** = <4, 3> and **w** = <2, –6>.

a. Find real numbers j and k such that j**v** + k**w** = <2, 14>.
(Give your answers as exact values.)

b. Suppose **v** and **w** are regarded as one-column matrices, $\mathbf{v} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$.

Find a 2-by-2 matrix M such that $M\,\mathbf{v} = \mathbf{w}$.

9. The moon goes through a cycle of phases from full (completely lit) to new (completely unlit) and back to full every 29 days. Consider the portion of the moon that is lit (when viewed from the earth as a circular disc) as a function of time. It is a fact that this function is approximately sinusoidal, so let t represent time in days and let $f(t)$ represent a sinusoidal function that models the portion of the moon that is lit. The range of $f(t)$ is from 0 (unlit) to 1 (fully lit). Further suppose that the moon is fully lit on when $t = 4.00$.

a. Write a function formula for sinusoidal function $f(t)$.

b. Using the $f(t)$ model, find the first 3 times (smallest 3 positive values of t) at which $\frac{3}{4}$ of the moon's disc will be lighted. (Give your answer as a decimal approximation accurate to the nearest 0.01.)

10. The equations of two planes are given below:

Plane P_1 : $2x + 3y + 5z = -2$

Plane P_2 : $5x + y - 7z = -18$

a. Find the vector equation of the line that is formed where the two planes intersect.

b. Find the acute angle formed by the two planes. (Give your answer as a decimal approximation accurate to the nearest 0.01 degree or 0.01 radian.)

11. Suppose $z = a + bi$ and $\overline{z} = a - bi$ (the conjugate of z). If $g(z) = \frac{3}{z}$, show that $g(\overline{z}) = \overline{g(z)}$.

12. For the following proof, the only trigonometric identities you may assume are:

- $\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$,
- all the Pythagorean identities.

Prove that $\sin^2(x) = \frac{1 - \cos(2x)}{2}$.

Part C. Long Problems

3 problems, 8 points each, 24 points total

13. Consider the linear transformation T of the plane described by this matrix equation:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Let E stand for the ellipse whose axes are the line segments from (1, 3) to (7, 3) and from (4, –2) to (4, 8).

In answers to the following questions, numbers may be given as decimal approximations accurate to the nearest 0.01.

a. Find the input point whose image under transformation T is (x', y') = (–2, 4).

b. Find the coordinates of the foci (focal points) of ellipse E.

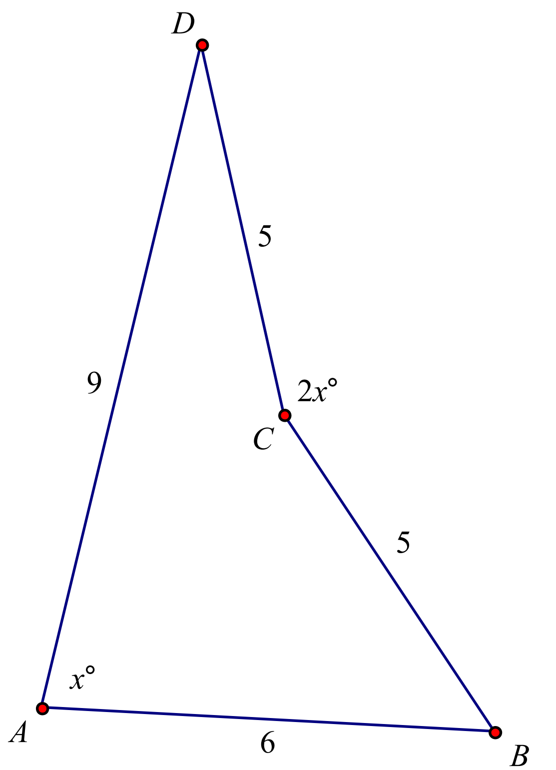
c. Write a pair of parametric equations describing ellipse E.

d. Suppose that transformation T is applied to ellipse E. The image will be another ellipse. Find a pair of parametric equations describing the image ellipse.

14. In quadrilateral *ABCD* (shown below), $AB = 6$, $BC = 5$, $CD = 5$, $AD = 9$, and the measure of $\angle DCB$ is twice the measure of $\angle DAB$.

You should not assume that the diagram is drawn exactly to scale, but do assume that the quadrilateral is concave, and that $\angle DCB$ is an obtuse angle.

Find the area of quadrilateral *ABCD*. (Give your answer as a decimal approximation accurate to the nearest 0.01.)



15. These facts are given about a triangle *ABC* in the plane:

- The rectangular coordinates of point *A* are $(x, y) = (1, 2)$.
- Vector *AB* = <4, 6>.
- Point *C* is in the 4th quadrant (lower right quadrant).
- Length $AC = 9$.
- Angle $BAC = \frac{2\pi}{5}$ radians.

In answering the following questions, you may give numbers as decimal approximations accurate to the nearest 0.01.

a. Find vector *AC* in component form.

b. Find the polar coordinates (r, θ) of point *C*.