

Level 1 Advanced Math 2005 Final Exam

NAME: ANSWERS AND GRADING GUIDELINES

Instructions

WRITE ANSWERS IN THE SPACES PROVIDED AND SHOW ALL WORK. Partial credit will not be given if work is not shown. Ask for extra paper if you need it. NOTHING ON THE EXTRA PAPER will be graded unless you explicitly write "see extra paper" on this exam paper AND you clearly indicate the problem number on the extra paper. Make sure your name is on any paper you want graded.

CALCULATORS are permitted. FORMULA SHEETS are attached at the end of the exam. You may detach these if you wish.

ALL PAPERS (the exam, extra paper, and the formula sheets) will be collected at the end of the exam.

POINT VALUES are marked on each problem. 100 points total

1. (5 points, 1 point per part) The definition of a piecewise function is given below. Answer the questions that follow.

$$f(x) = \begin{cases} 2x - 3 & \text{if } x < -1 \\ |x| - 5 & \text{if } -1 \leq x \leq 2 \\ x^2 & \text{if } 2 < x < 2\pi \end{cases}$$

Find:

a. $f(0)$ -5 b. $f(1.6)$ -3.4 c. $f(-2.3)$ -7.6

d. $f(2)$ -3 e. The domain of f: $-\infty < x < 2\pi$

1 POINT PER PART; NO PARTIAL CREDIT

2. (2 points) State the domain of the function $f(x) = \sqrt{x+1} + \frac{x+2}{x-2}$

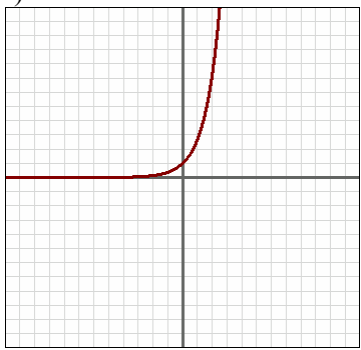
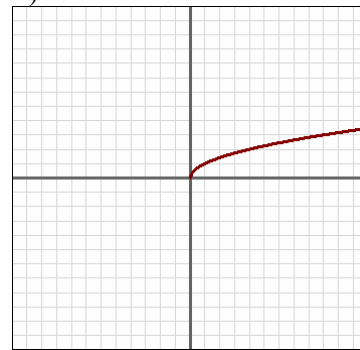
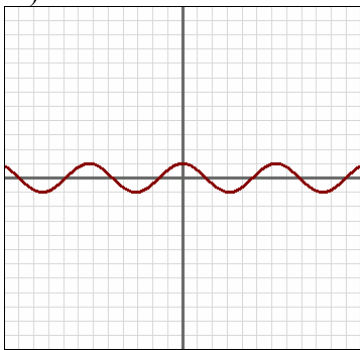
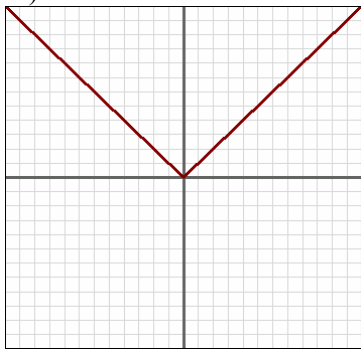
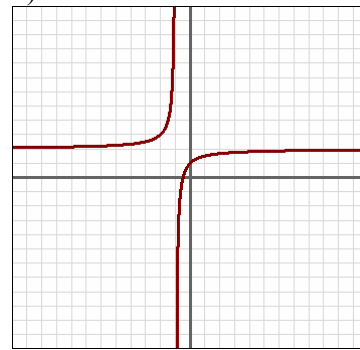
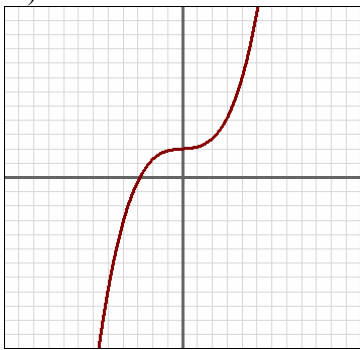
Domain: $x \geq -1$ and $x \neq 2$

ONE POINT FOR EACH OF THE TWO PARTS OF THE ANSWER

3. (9 points, 1 points for each of 9 parts) Below are 12 algebraic formulas and nine graphs of some of the functions we studied this year. For each graph, state which of the formulas (A—L) best matches the graph. You will not use all of the formulas. THE GRAPHS CONTINUE ON THE NEXT PAGE.

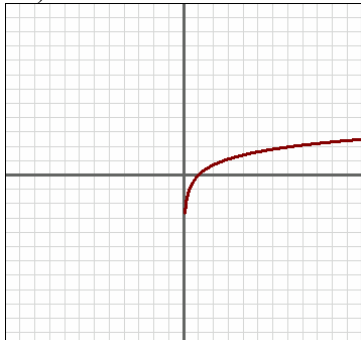
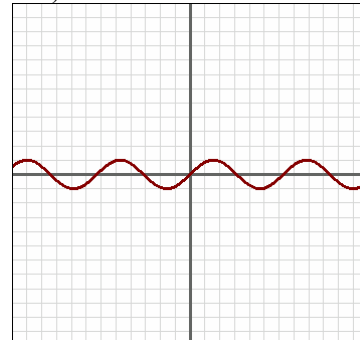
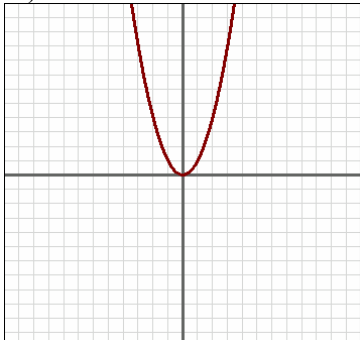
ONE POINT PER PART.

A) $f(x) = a(x - h)^2 + k$	B) $f(x) = A x - C + D$	C) $f(x) = \frac{Ax + B}{Cx + D}$
D) $f(x) = A \sin Bx$	E) $f(x) = A \cos Bx$	F) $f(x) = A \tan Bx$
G) $f(x) = mx + b$	H) $f(x) = AB^{kx} + D$	I) $f(x) = A \log_B(x + C) + D$
J) $f(x) = A\sqrt{x + C} + D$	K) $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$	L) $f(x) = A[B(x + C)] + D$ $f(x) = A[B(x + C)] + D$ (These are alternate notations for the same function)

i) 	ii) 	iii) 
a) Fomula _____ H _____	a) Fomula _____ J _____	a) Fomula _____ E _____
iv) 	v) 	vi) 
a) Fomula _____ B _____	a) Fomula _____ C _____	a) Fomula _____ K _____

(Formulas repeated for reference)

A) $f(x) = a(x - h)^2 + k$	B) $f(x) = A x - C + D$	C) $f(x) = \frac{Ax + B}{Cx + D}$
D) $f(x) = A\sin Bx$	E) $f(x) = A\cos Bx$	F) $f(x) = A\tan Bx$
G) $f(x) = mx + b$	H) $f(x) = AB^{kx} + D$	I) $f(x) = A\log_B(x + C) + D$
J) $f(x) = A\sqrt{(x + C)} + D$	K) $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$	L) $f(x) = A[B(x + C)] + D$ $f(x) = A[B(x + C)] + D$ (These are alternate notations for the same function)

vii)	viii)	ix)
		
a) Formula _____ I _____	a) Formula _____ D _____	a) Formula _____ A _____

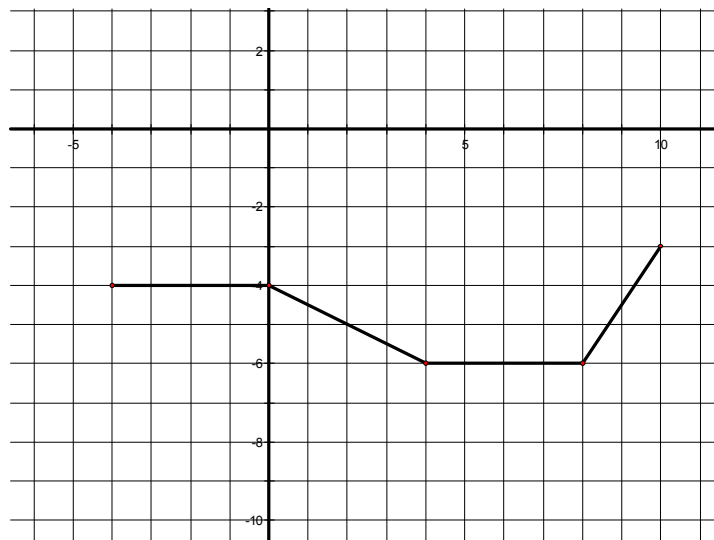
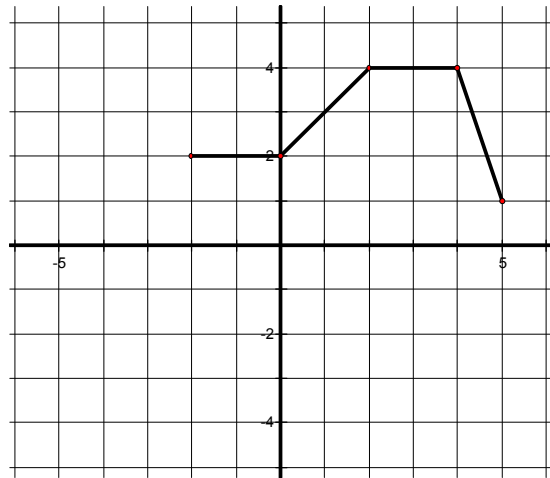
4. (4 points, 2 points per part) Let $f(x) = \sqrt{3x + 7}$

a. Compute the inverse function $f^{-1}(x)$ _____ $f^{-1}(x) = \frac{x^2 - 7}{3}$ for $x \geq 0$ _____

TWO POINTS PER PART; NO PARTIAL CREDIT (STUDENTS NEED NOT SPECIFY THE RESTRICTED DOMAIN OF THE INVERSE FUNCTION).

b. Compute $(f^{-1} \circ f)(x)$: _____ = x _____

5. (3 points) The graph of a function $f(x)$ is given below. Sketch a graph of the transformation $g(x) = -f\left(\frac{1}{2}x\right) - 2$ on the empty grid below.



ONE POINT FOR EACH OF THE THREE FEATURES OF THE TRANSFORMATION:

- HORIZONTAL STRETCH BY 2
- REFLECT OVER X AXIS
- SHIFT DOWN 2

6. (2 points) Let $h(x) = \sqrt{3x^2 + 1}$. Write $h(x)$ as a composition of two functions.
 $h(x) = (f \circ g)(x)$ THE IDENTITY FUNCTION $f(x) = x$ MAY NOT BE PART OF YOUR ANSWER. (Hint: there is more than one correct answer).

$$g(x) = \underline{\hspace{2cm}} 3x^2 + 1 \underline{\hspace{2cm}} \qquad f(x) = \underline{\hspace{2cm}} \sqrt{x} \underline{\hspace{2cm}}$$

THERE IS MORE THAN ONE CORRECT ANSWER

ONE POINT FOR EACH OF THE TWO FUNCTIONS.

0 POINTS FOR USING THE IDENTITY FUNCTION

-1 POINT FOR CORRECT FUNCTIONS IN WRONG ORDER

7. (6 points) The functions $f(x)$ and $g(x)$ are defined in the first three columns of the table below. Complete the fourth and fifth columns. If a function is undefined for a given input, then write "undefined" in the table.

x	$f(x)$	$g(x)$	$(f \circ g)(x)$	$(f \div g)(x)$
0	3	2	1	3/2 OR 1.5
1	5	0	3	undefined
2	1	3	undefined	1/3 OR .33

ONE POINT FOR EACH OF THE 6 ANSWERS

8. (3 points) Find the zeros of this function. You may give your answers in radical form (like $5\sqrt{7}$) or decimal form to the nearest hundredth (like 13.23). If there are no zeros, write "no zeros"

$$f(x) = x^4 - 5x^2 + 6 \qquad \text{Zeros: } \underline{\hspace{2cm}} \pm \sqrt{2}, \pm \sqrt{3} \quad \text{OR} \quad \pm 1.41, \pm 1.73 \underline{\hspace{2cm}}$$

ALL FOUR ZEROS: 3 POINTS

TWO OR THREE: 2 POINTS

ONE: 1 POINT

9. (3 points) What is the remainder when $x^4 + 3x^3 - 11x^2 - 3x + 10$ is divided by $(x - 2)$?

$$\text{Remainder: } \underline{\hspace{2cm}} 0 \underline{\hspace{2cm}}$$

CORRECT METHOD (DIVISION OR SUBSTITUTION) BUT WRONG ANSWER DUE TO CALCULATION ERROR: 2 POINTS

WRONG ANSWER AND NO WORK SHOWN: 0 POINTS

10. (2 points) Which rational function below has the following features: asymptotes $x = -2$ and $y = 1$ and a hole at $x = 3$?

a. $y = \frac{(x-3)(x+3)}{(x+2)(x-3)}$

b. $y = \frac{(x-3)}{(x+2)(x-3)}$

c. $y = \frac{(x-3)(x+3)}{(x-2)(x-3)}$

ANSWER: a

TWO POINTS OR NONE

11. (2 points) Simplify: $e^{2\ln x}$

Answer: _____ x^2 _____

2 POINTS OR NONE; NO PARTIAL CREDIT

12. (3 points) Solve for x: $\log x + \log(x+3) = 1$

$x =$: _____ $x = 2$ _____

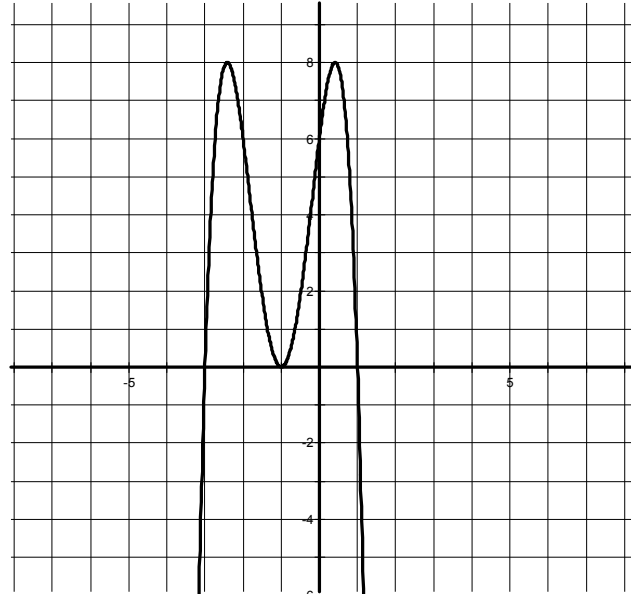
NOTE: $x = -5$ IS AN EXTRANEIOUS SOLUTION

INCLUDING $x = -5$ AS AN ANSWER: -1 POINT

ANY CORRECT APPLICATION OF A LAW OF LOGARITHMS: +1 POINT

EXPONENTIATING BOTH SIDES OF THE EQUATION: +1 POINT

13. (8 points, as marked) The graph of a polynomial function is shown at right.



a. (1 point) Is the leading coefficient positive or negative?

_____NEGATIVE_____

b. (1 point) Is the degree even or odd?

_____EVEN_____

c. (2 points) What is the smallest possible degree of the polynomial? Give a reason for your answer.

Degree: _____4_____

Reason: _____SEVERAL POSSIBLE ANSWERS:

3 TURNING POINTS _____

4 ZEROS (COUNTING MULTIPLES)

3 ZEROS AND EVEN _DEGREE_

d. (2 points) What are the zeros, and what is the smallest possible multiplicity of each?

_____ -3, mult 1; -1, mult 2; +1, mult 1 _____

e. (2 points) Write a lowest-degree polynomial that has the graph in the previous problem. Note that $f(0) = 6$. (You may write in factored form)

_____ $f(x) = -2(x+3)(x+1)^2(x-1)$ _____

PART C: ONE POINT FOR CORRECT DEGREE, ONE FOR CORRECT REASON

PART D: TWO POINT FOR ALL 6 VALUES (3 ZEROS, 3 MULTIPLICITIES)

ONE POINT FOR AT LEAST 3 VALUES; NO POINTS FOR LESS THAN 3

PART E: TWO POINTS FOR ALL 3 FACTORS, THE EXPONENTS, AND LEADING COEFFICIENT. ONE POINT FOR 3 FACTORS. ZERO OTHERWISE.

14. (4 points, 2 per part) You deposit \$5,000 at a bank that pays 7% interest. What is the balance in your account if the interest is compounded as follows: (SHOW THE CORRECT FORMULA AS PART OF YOUR ANSWER)

a.) Quarterly for 10 years (to the nearest \$0.01): _____ \$10,007.99 _____

$$A = P \left(1 + \frac{r}{n} \right)^{nt} = 5000 \left(1 + \frac{.07}{4} \right)^{40}$$

ONE POINT EACH FOR CORRECT FORMULA AND ANSWER. FORMULA MAY BE EITHER THE GENERAL VERSION (LETTER VALUES) OR THE SPECIFIC VERSION FOR THIS PROBLEM

b.) Continuously for 10 years (to the nearest \$0.01): _____ \$10,068.76 _____

$$A = Pe^{rt} = 5000 \cdot e^{.07 \cdot 10}$$

SAME GUIDELINES AS PART a.

15. (5 points, as marked) For the function $y = \frac{2x-6}{x-5}$

a. (2 points) Write the equations for any horizontal and vertical asymptotes.

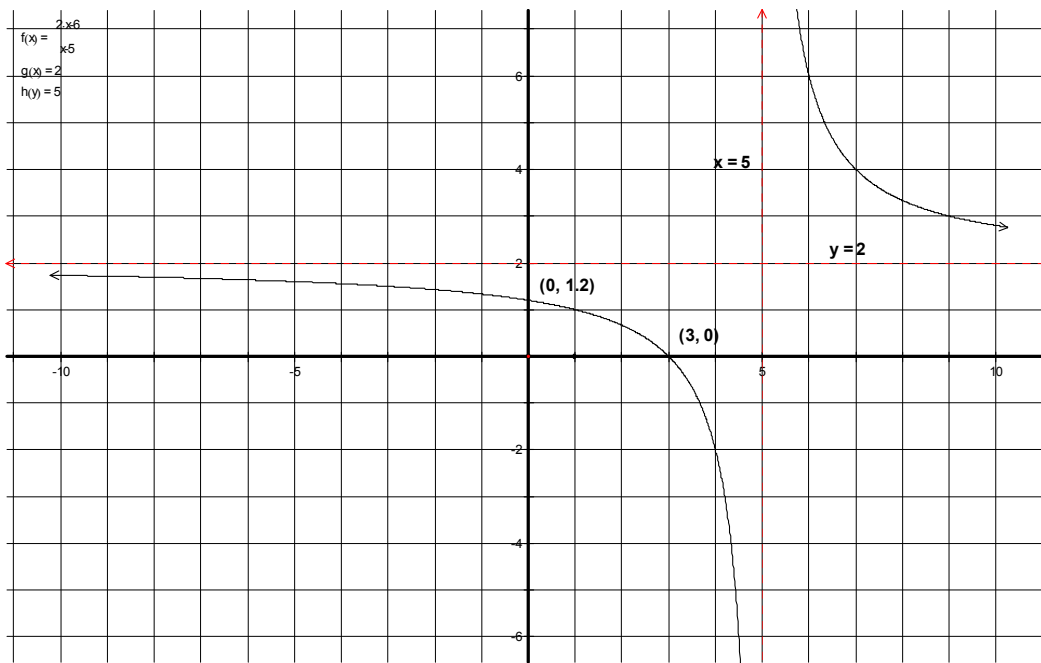
_____ Vertical asymptote: $x = 5$, horizontal asymptote: $y = 2$ _____

ONE POINT PER ASYMPTOTE

b. (3 points) Graph the function. The graph should include all vertical and horizontal asymptotes, zeros, holes, and y-intercepts (if they exist). Show the asymptotes as dotted lines. Label the zeros, holes, and y-intercept with (x,y) coordinates.

SCALE: 1 BLOCK = 1 UNIT

(NOTE: the asymptotes may not display and/or print properly in this Sketchpad drawing.)



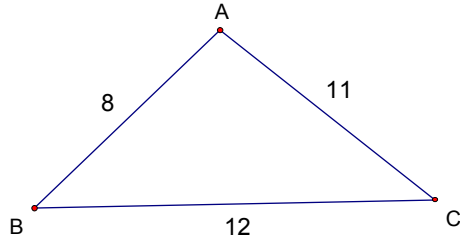
ASYMPTOTES AS IN PART A; X-INTERCEPT 3, Y-INTERCEPT 6/5; NO HOLES. GRAPH SHOULD APPROACH $y = 2$ FROM BELOW ON THE LEFT AND FROM ABOVE ON THE RIGHT. SHOULD APPROACH $-\infty$ TO LEFT OF $x=5$ AND $+\infty$ TO RIGHT

3 POINTS FOR ALL FEATURES CORRECT. 1 POINT FOR THE GENERAL SHAPE.

0 POINTS IF STUDENT JUST COPIES THE CALCULATOR SCREEN, WITH NO ASYMPTOTES, LABELS, ETC.

CALCULATORS IN DEGREE MODE—CHECK NOW!

16. (3 points, 1 per part) Solve triangle ABC. Express your answers to the nearest hundredth.



$$\angle A = \underline{\quad 76.53^\circ \quad}$$

$$\angle B = \underline{\quad 63.06^\circ \quad}$$

$$\angle C = \underline{\quad 40.42^\circ \quad}$$

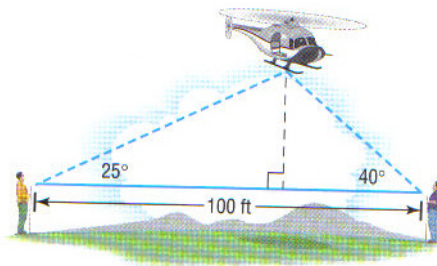
RADIAN ANSWERS ARE 1.34, 1.10, 0.71. -1 POINT

1 POINT PER ANSWER; FORMULAS/CALCULATIONS NOT REQUIRED.

1 POINT IF NO CORRECT ANSWERS BUT CORRECT FORMULA (LAW COSINES).

17. (3 points) Two observers simultaneously measure the angle of elevation of a helicopter. One angle is measured as 25° , the other as 40° . (See figure below). If the observers are 100 feet apart and the helicopter lies over the line joining them, how high is the helicopter? Express your answer to the nearest hundredth.

Height: 29.97 ft



CORRECT FORMULAS (THERE ARE AT LEAST 2 WAYS TO DO THIS): 2 POINTS

METHOD 1:

Angle at helicopter is 115

Slant distance from observer on left to helicopter is $\frac{100 \cdot \sin 40}{\sin 115} = 70.923$

Height of helicopter is $70.925 \sin 25 = 29.97$

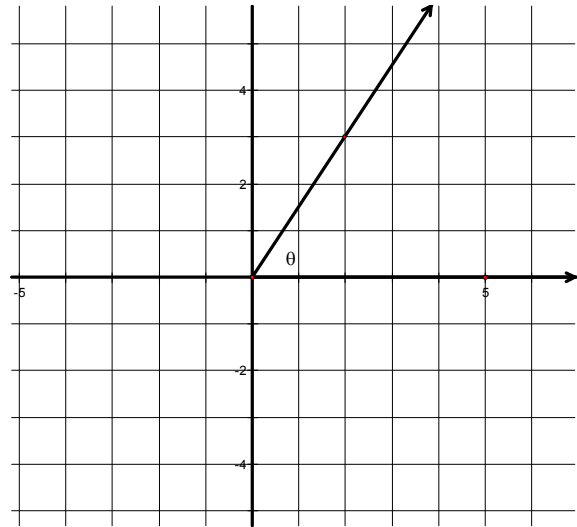
CALCULATORS IN RADIAN MODE—CHECK NOW!

18. (3 points) The terminal side of angle θ passes through the point (2, 3). What is the measure of angle θ (in radians)? Express your answer to the nearest hundredth.

_____ 0.98 ____ (.31 π) _____

DEGREE ANSWER: 56.31° -1 POINT

CORRECT FORMULA INVOLVING AN INVERSE TRIG FUNCTION (THERE IS MORE THAN ONE): 2 POINTS



19. (3 points) Prove the identity $\frac{\tan(x)}{1 - \cos^2(x)} = \csc(x) \cdot \sec(x)$

CORRECT SUBSTITUTIONS FOR tan, csc, sec: ONE POINT

THERE ARE SEVERAL METHODS TO PROVE THE IDENTITY.

ONE METHOD:

$$\frac{\tan(x)}{1 - \cos^2(x)} = \frac{\frac{\sin(x)}{\cos(x)}}{\sin^2(x)} = \frac{\sin(x)}{\cos(x)} \cdot \frac{1}{\sin^2(x)} = \frac{1}{\cos(x)} \cdot \frac{1}{\sin(x)} = \sec(x) \cdot \csc(x)$$

20. (3 points) Find all solutions of the equation $\sin(2x) = \frac{\sqrt{2}}{2}$ (in radians) Express your answers to the nearest hundredth.

$$x = \text{_____} \quad x = \left\{ \begin{array}{l} 0.3927 + k\pi \\ 1.1781 + k\pi \end{array} \right\} \text{---OR---} \quad x = \left\{ \begin{array}{l} \frac{\pi}{8} + k\pi \\ \frac{3\pi}{8} + k\pi \end{array} \right\} \text{_____}$$

ONLY ONE SET OF SOLUTIONS: -1 POINT

WRONG PERIODIC TERM (E.G., $2k\pi$): -1 POINT

21. (4 points) Write the sinusoidal function represented by this graph.:

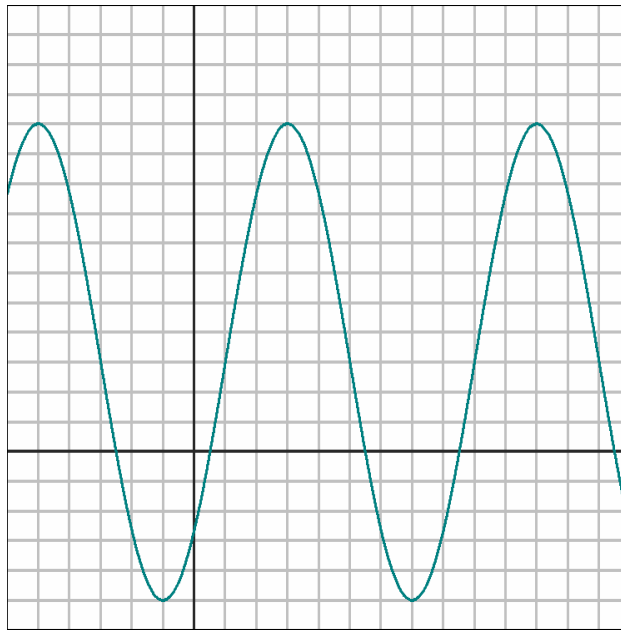
_____ $y = 3 + 8 \cos \frac{\pi}{4}(x - 3)$ or $y = 3 + 8 \sin \frac{\pi}{4}(x - 1)$ _____

PHASE DISPLACEMENTS A MULTIPLE OF 8 DIFFERENT FROM THESE ARE ALSO CORRECT.

ONE POINT PER PARAMETER (A, B, C, D)

Horizontal Axis: SCALE 1 BLOCK = 1 RADIAN

Vertical Axis: SCALE: 1 BLOCK = 1 UNIT



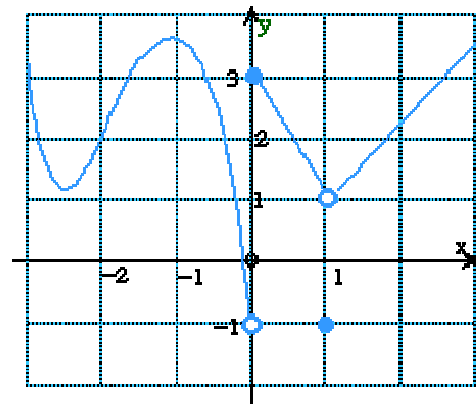
22. (4 points, 1 point each part) Use the graph of $f(x)$ at the right to identify the following limits. If a limit is not a finite number, then give one of the following answers ∞ , $-\infty$. If no limit exists, write “no limit”.

a) $\lim_{x \rightarrow 0^+} f(x) =$ _____ 3 _____

b) $\lim_{x \rightarrow 0} f(x) =$ _____ no limit (OR does not exist) _____

c) $\lim_{x \rightarrow 1} f(x) =$ _____ 1 _____

d) $\lim_{x \rightarrow +\infty} f(x) =$ _____ ∞ _____



ONE POINT PER PART, NO PARTIAL CREDIT

23. (6 points, 3 per part) In a tidal river, the depth of the water varies sinusoidally as a function of time. The time between high tide and low tide is 6 hours (therefore, a full cycle is 12 hours). At high tide the depth of the water at a certain dock is 18 feet, while at low tide the depth of the water is 6 feet. High tide occurs at 12:30 PM. Let $d(t)$ represent the depth of the water in feet as a function of t , hours after 12 noon.

a) Write an equation for the function $d(t)$. Clearly show how you determined the 4 constant values in your equation.

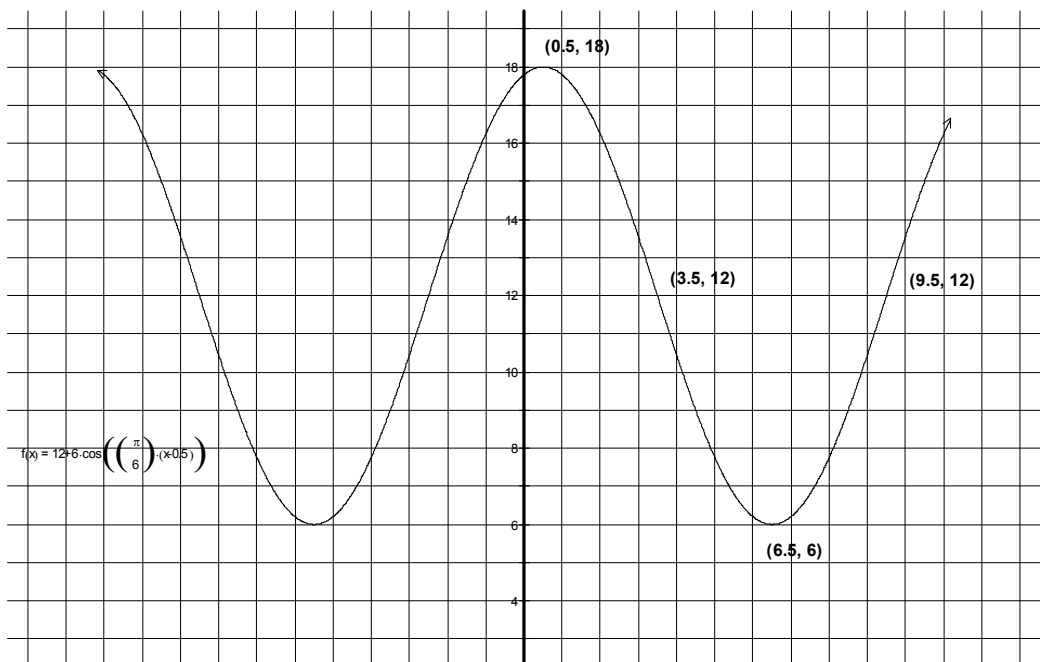
$$y = 12 + 6 \cos \frac{\pi}{6}(t - 0.5) \text{ OR } y = 12 + 6 \sin \frac{\pi}{6}(t + 2.5)$$

OK TO HAVE 0.5236 IN PLACE OF $\pi/6$.

PHASE DISPLACEMENTS DIFFERENT BY A MULTIPLE OF 12 ARE ALSO CORRECT.

-1 POINT PER INCORRECT PARAMETER (A, B, C, D) UP TO 3 POINTS MAXIMUM

b) Sketch a graph of the depth of the water over time on the grid below. Show at least one full cycle of the function. Clearly label the scale of your axes. CLEARLY LABEL (WITH (x, y) COORDINATES) THE “CRITICAL POINTS” IN THIS FULL CYCLE.



FULL CREDIT FOR A CORRECT GRAPH OF AN INCORRECT ANSWER TO PART a.

NO COORDINATES ON CRITICAL POINTS: -1 POINT

PERIOD WRONG: -1 POINT; AMPLITUDE WRONG: -1 POINT, VERT DISPLACEMENT

WRONG: -1 POINT

24. (3 points) How many ways can 2 goats, 3 cows, and 4 pigs be lined up so that animals of the same species are all next to each other? SHOW BOTH FORMULAS AND THE FINAL ANSWER.

Number of ways: _____ 1728 _____

Arrange the 3 species: 3!

Arrange the animals within each species: 2!3!4!

Total: 3!2!3!4! = 1728

ONE OF THE FOUR FACTORS MISSING: TWO POINTS

25. (4 points, 2 per part). Assume that enrollment at LHS is 500 each of freshmen, sophomores, juniors, and seniors (and assume the seniors are still here). (SHOW BOTH FORMULAS AND THE FINAL ANSWER).

a. If 4 students are selected at random, what is the probability they are 1 of each class? (Express your answer to the nearest hundredth.)

Probability: _____ .094 _____

$$\frac{500 C_1 \cdot 500 C_1 \cdot 500 C_1 \cdot 500 C_1}{2000 C_4} \quad \text{OR} \quad 1 \cdot \frac{1500}{1999} \cdot \frac{1000}{1998} \cdot \frac{500}{1997}$$

Answer of $\left(\frac{1}{4}\right)^4$: 1 POINT

b. If 4 are selected at random, what is the probability that none is a freshman? (Express your answer to the nearest hundredth.)

Probability: _____ .32 _____

$$\frac{1500}{2000} \cdot \frac{1499}{1999} \cdot \frac{1498}{1998} \cdot \frac{1497}{1997} \quad \text{OR} \quad \frac{1500 C_4}{2000 C_4}$$

26. (3 points) (DEGREE MODE) You are standing on the roof of a building ~~50 feet tall~~ (see note). There is a taller building 150 feet across the street. The angle of elevation from you to the top of the other building is 42° . The angle of depression from you to the base of the building is 30° . How tall is the other building (Express your answer to the nearest hundredth.)? (INCLUDE A DIAGRAM AS PART OF YOUR SOLUTION)

Height of building: _____ 221.66 _____

NOTE:

$$150 \tan(42^\circ) + 150 \tan(30^\circ)$$

NOTE: THERE IS AN ERROR IN THE PROBLEM STATEMENT. THE HEIGHT OF THE FIRST BUILDING (50 FT) IS INCONSISTENT WITH THE OTHER DATA ITEMS, AND SHOULD HAVE BEEN OMITTED. BECAUSE OF THE ERROR, THE VALUES 185.06 AND 214.11 WERE ALSO ACCEPTED AS CORRECT ANSWERS TO THE QUESTION.