Honors Calculus AB Name________________

Final Exam: Open Response 5/5/2003

Answer 3 of the following 5 open response questions. Make sure to write complete solutions to your problems in order to receive partial credit. Each of the 3 questions you answer will be weighted equally. List the 3 problems you want graded on the lines below. Good luck!

I would like problems _______ , ________ , and ___________ graded.

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1. Let R denote the region bounded by the x-axis, $x = 2$, and the curve $y = x^2$.

a) Draw the region R on the axes provided below.

b) Find the area of region R

c) Find the volume of the solid formed when region R is revolved around the x-axis.

d) Find the volume of the solid formed when the region R is revolved around the y-axis.

Solution

a) The graph should have the appropriate bounds.

b) $\int_0^2 x^2 dx = 8/3$

c) $\int_0^2 \pi (x^2)^2 dx = 32\pi/5 \approx 20.106$

d) Washers $\int_0^4 \pi(2^2 - \sqrt{y}^2)dy = 8\pi \approx 25.132$

Shells $\int_0^2 2\pi x(x^2)dx = 8\pi \approx 25.132$

2. A particle travels along the x-axis with acceleration $a(t) = -4t$. Initially, time $t = 0$, the particle is located at the origin and has velocity –4 units/second.

a) Find the velocity *v(t)* and position *s(t)* functions.
b) What is the maximum *speed* of the particle over the interval [0,3] and at what time does it occur?
c) What is the total distance the particle travels in the first 3 seconds of its motion?

Solution

a) For the velocity equation

$$v(t) = \int a(t) = \int -4t\,dt = -2t^2 + c$$

constant of integration:

$$v(0) = -2(0)^2 + c = -4$$

$$\Rightarrow c = -4$$

so, $v(t) = -2t^2 - 4$

For the position equation

$$s(t) = \int v(t) = \int (-2t^2 - 4)dt = -2/3t^3 - 4t + c$$

constant of integration:

$$s(0) = -2/3(0)^3 - 4(0) + c = 0$$

$$\Rightarrow c = 0$$

so, $s(t) = -2/3t^3 - 4t$

b) $v(t)$ is decreasing on [0,3], so largest value (in absolute) occurs at endpoint of interval $t = 3$. speed = $|v(3)| = |-22| = 22$ units/s

c) total distance = $|\text{displacement}| = \int_0^3 |v(t)| = \int_0^3 |-2t^2 - 4|dt = 30$ units

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3. Let $f(x) = 1 + \sqrt{x+3}$.

a) Find the *average* rate of change of *f* on the interval [-3,1].
b) Find an *x*-value, *c*, such that the *instantaneous* rate of change at *c* is equal to the *average* rate of change on the interval [-3,1].
c) Find the equation of the line tangent to *f* that passes through *c*.
d) Parts a-c of this problem relate to the *Mean Value Theorem*. State this theorem and illustrate its geometric meaning on the graph of the function provided below.

Solution

a) average rate of change = $\frac{f(1)-f(-3)}{1-(-3)} = 1/2$

b) $\text{instantaneous rate of change} = \text{average rate of change}$

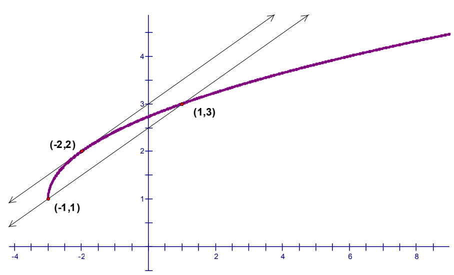
$$f'(x) = 1/2$$

$$\tfrac{1}{2}(x+3)^{-1/2} = 1/2$$

$\Rightarrow x = -2$, so the i.r.o.c. equals the a.r.o.c. at c = - 2.

c) When x = -2, y = 2, so $y - 2 = 1/2(x+2)$

d) On a closed interval [a,b], for a continuous function, *f*, there must exist a point, *c*, such that $f'(c) = \frac{f(b)-f(a)}{b-a}$. Graphically, there must exist a $m_{\tan gent} = m_{\sec ant}$.



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4. Given the curve $f(x) = -(x-3)^2 + 4$

   a) Find the average value of the function over the interval [1,4].
   b) Use calculus to explain why in the interval [2,4] there must exist an x-value, c, such that $f'(c) = 0$.
   c) Use calculus to explain why in the interval [3,6] the graph of the function f must cross the x-axis exactly once.

Solution

a) $av_f = \frac{1}{4-1}\int_1^4 (-(x-3)^2 + 4)dx = 3$

b) Notice that $f(2) = f(4) = 3$. The average rate of change between 2 and 4 is 0, so, by the Mean Value Theorem (see 3d), there exist a value c such that $f'(c) = 0$.

c) Notice that $f(3) = (+)$ and $f(6) = (-)$. Since f is a continuous function, it must pass though all y values between $f(3)$ and $f(6)$, including 0. Since f is decreasing on the entire interval [3,6], there are no critical points on that interval and it can not possibly "double back" to cross the x-axis twice.

To show that f is decreasing on [3,6], you could use:

i. first derivative test + concavity

ii. properties of quadratic equations (not as good)

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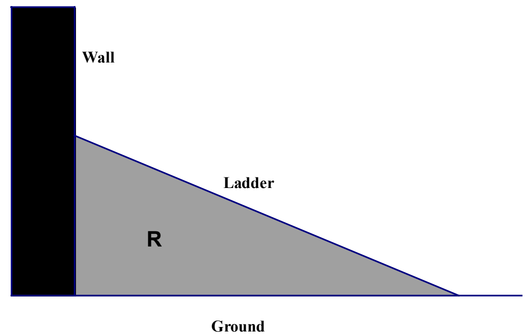
5. A 13-foot ladder is leaning against a house when its base starts to slide away. By the time the base is 12 feet from the house, the base is moving at the rate of 5 ft/sec.

   a) How fast is the top of the ladder sliding down the wall at that moment when the base is 12 feet from the house?



Solution

$x^2 + y^2 = 13$

$\Rightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

$\Rightarrow 2(12)(5) + 2(5)\frac{dy}{dt} = 0$

$\Rightarrow \frac{dy}{dt} = -12 \text{ ft/sec}$

   (b) At what rate is the area of the triangular region **R** changing at that moment?

Solution

$A(x) = 1/2xy = .5(x\frac{dy}{dt} + \frac{dx}{dt}y) = 1/2[12(-12) + 5(5)] = -59.5 \text{ ft}^2\text{/sec}$

An alternate interpretation could yield:

$A(x) = 1/2xy = 1/2x\sqrt{169 - x^2} \Rightarrow \frac{dA}{dx}|_{x=12} = -11.9 \text{ ft}^2\text{/ft}$

If you understand the difference between these two responses you understand this concept.

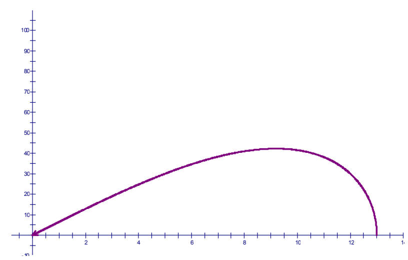
   (c) Make an accurate graph of *Area of* **R** versus *Distance from wall* for distances between 0 and 13 feet.

Solution

Graph should be the graph of the equation:

$A(x) = 1/2xy = 1/2x\sqrt{169 - x^2}$,

which is positive from 0 to 13.



   (d) Where is the area of **R** greatest? Justify your response.

Solution

Two analytic solutions to this part:

i. Find where $\frac{dA}{dx} = 0$ (kind of hard)

ii. Notice that the area will be at a max when the triangle formed is isosceles right.

iii. Find the max using graphing calculator utility.

These methods all yield: when $x = 9.19$ ft., the area is at a maximum (42.25ft^2). A good graph in part C) should reinforce this value.