

### Part A. Complex number system

- Since  $w = a + bi$  is in the 4th quadrant,  $a$  is positive and  $b$  is negative.
  - $\bar{w} = a - bi$ . Since  $a$  is positive and  $-b$  is positive,  $\bar{w}$  lies in the 1st quadrant.
  - $-w = -a - bi$ . Since  $-a$  is negative and  $-b$  is positive,  $-w$  lies in the 2nd quadrant.
  - $\frac{1}{w} = \frac{1}{a+bi} = \frac{(a-bi)}{(a+bi)(a-bi)} = \frac{(a-bi)}{a^2+b^2} = \frac{a}{a^2+b^2} + \frac{-b}{a^2+b^2}i$ . Since  $\frac{a}{a^2+b^2}$  is positive and  $\frac{-b}{a^2+b^2}$  is positive,  $\frac{1}{w}$  lies in the 1st quadrant.
- $z^3 = (a + bi)(a + bi)(a + bi) = (a^2 + 2abi - b^2)(a + bi) = a^3 + 3a^2bi - 3ab^2 - b^3i$ .  
 $(\bar{z})^3 = (a - bi)(a - bi)(a - bi) = (a^2 - 2abi - b^2)(a - bi) = a^3 - 3a^2bi - 3ab^2 + b^3i$ .  
 Conclude that  $z^3$  and  $(\bar{z})^3$  are conjugates.

### Part B. Polynomials

- The factorization in the complex number system is  $P(x) = (x + 1)^2(x - 1)(x - (2 - i))(x + (2 + i))$ .  
 Multiply the last two factors:  $(x - (2 - i))(x - (2 + i)) = x^2 - 4x + 5$ .  
 So the factorization in the real number system is  $P(x) = (x + 1)^2(x - 1)(x^2 - 4x + 5)$ .
- First write  $z^3 - 8 = 0$ . Since  $z = 2$  is a zero,  $(z - 2)$  is a factor. Long division shows that  $(z^3 - 8)/(z - 2) = (z^2 + 2z + 4)$ . So we have  $(z - 2)(z^2 + 2z + 4) = 0$ . Now use the quadratic formula to get that the zeroes of the quadratic are  $z = -1 \pm \sqrt{3}i$ .  
 Solution set:  $\{2, -1 + \sqrt{3}i, -1 - \sqrt{3}i\}$ .
- The assertion about the zero in the interval seems to be an attempt to use the Intermediate Value Theorem. However, the Intermediate Value Theorem applies only when a function is continuous throughout an interval. Since  $f(x)$  is a rational function, it might have discontinuities in the interval, so this theorem may not be applied to  $f(x)$ .
  - It is possible that  $C(x)$  could have only the single real zero and no non-real zeroes. The use of the Fundamental Theorem of Algebra was incorrect. That theorem does say that  $C(x)$  has three complex zeroes, but only when counted with multiplicity. It is possible for  $C(x)$  to have as its only zero be a real number with multiplicity 3. For example,  $C(x) = (x-1)^3$  has  $x = 1$  as a zero with multiplicity 3, and no other zeroes.

### Part C. Rational functions

This factorization step will be useful:  $R(x) = \frac{3x^3 + 6x^2 - 9x}{x^2 + x} = \frac{3x(x+3)(x-1)}{x(x+1)}$ .

- $R(x)$  has discontinuities at the zeros of the denominator,  $x = -1$  and  $x = 0$ .  
 Since the  $(x + 1)$  factor does not cancel, there is a vertical asymptote at  $x = -1$ .  
 Since the  $x$  factors do cancel, there is a hole at  $x = 0$ .

2.

	$x < -3$	$-3 < x < -1$	$-1 < x < 0$	$0 < x < 1$	$1 < x$
numer.	-	+	+	-	+
demon.	+	+	-	+	+
$R(x)$	-	+	-	-	+

So  $R(x) > 0$  when  $-3 < x < -1$  or  $1 < x$ .

- Divide  $3x^3 + 6x^2 - 9x$  by  $x^2 + x$ . The quotient is  $3x + 3$ , so the end behavior polynomial (i.e., the asymptote) is  $P(x) = 3x + 3$ .