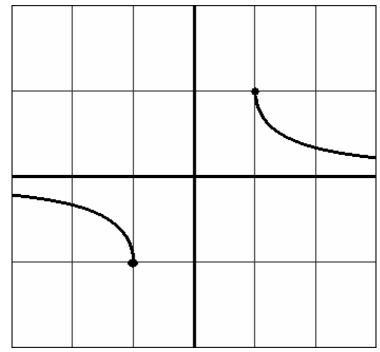
Part A. Inverse trigonometric functions

1. a. The original function csc x is not one-to-one (does not pass the horizontal line test). This means that if its inverse were formed, there would not be just one output value for each input value, so the inverse would not be a function. When csc x is restricted to an interval on which the function is one-to-one, then the inverse will be a function.

b.

restricted csc x	$\csc^{-1}(x)$
domain: $[-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$	domain: $(-\infty, -1] \cup [1, \infty)$
range: $(-\infty, -1] \cup [1, \infty)$	range: $[-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$



c. Graph of $\csc^{-1}(x)$ shown for the window $[-3, 3]$ by $[-\pi, \pi]$.
The endpoints are $(-1, -\pi/2)$ and $(1, \pi/2)$.
The graph has the x-axis as an asymptote.

2. a. Let $\theta = \csc^{-1}(-2)$.
Then $\csc(\theta) = -2$, so $\sin(\theta) = -1/2$, so $\theta = -\pi/6$.

b. $\cos(\sin^{-1} x) = \sqrt{1-x^2}$, based on the triangle shown in *Pre-Calculus* p. 400 Example 5.

Part B. Measurements related to triangles

1. a. Use "SAS" area formula. Answer: area ≈ 7.053 square units.

b. Use Law of Cosines. Answer: $XZ \approx 3.629$.

c. Call the angle bisector segment XP, where P is on side YZ.
Law of Cosines on ΔXYZ gives $\angle YXZ \approx 1.808$ radians.
The angle bisector gives that $\angle YXP \approx 0.909$ radians.
Then the third angle in ΔXYP is $\angle YPX \approx 1.604$ radians.
Finally Law of Sines on ΔXYP gives $XP \approx 2.352$.

An alternate approach would be to use the method of *Advanced Mathematics* p. 354 Exercises 21 and 24.

2. See *Advanced Mathematics* p. 349 Exercise 23 for an outline of the proof.
Part **a** uses that $\angle P$ and $\angle C$ are inscribed angles, so their measures are $\frac{1}{2}$ the measure of arc AB.
Part **b** uses that ΔABP is a right triangle.

Part C. Your choice

1. Draw the angle bisector of the angle, which passes through the centers of the circles.
Draw a radius from the center of each circle to a point of tangency with the angle.
These form two right triangles, each having an angle of measure x.
Let c = the length of the hypotenuse of the smaller triangle.

Applying the right triangle definition of sine to the two triangles gives

$$\sin x = \frac{r}{h} \quad \text{and} \quad \sin x = \frac{R}{h+r+R}.$$

Transform the equations into

$$h \sin x = r \quad \text{and} \quad h \sin x = R - r \sin x - R \sin x.$$

Transitivity gives $r = R - r \sin x - R \sin x$.
Collecting like terms and factoring gives $r(1 + \sin x) = R(1 - \sin x)$.
Finally, divide to get the desired conclusion.

2. From the given equation, multiply to get $c^2a + c^2b + c^3 = a^3 + b^3 + c^3$.
Simplify the equation to $c^2(a + b) = a^3 + b^3$.
Make a Law of Cosines substitution for c^2: $(a^2 + b^2 - 2ab \cos C)(a + b) = a^3 + b^3$.
Distribute: $a^3 + a^2b + b^2a + b^3 - 2a^2b \cos C - 2ab^2 \cos C = a^3 + b^3$.
Simplify: $a^2b + b^2a - 2a^2b \cos C - 2ab^2 \cos C = 0$.
Divide to get: $a + b - 2a \cos C - 2b \cos C = 0$.
Add to get: $a + b = 2a \cos C + 2b \cos C$.
Factor: $a + b = (2a + 2b) \cos C$.
So $\cos C = \frac{a+b}{2a+2b} = \frac{1}{2}$, giving that $\angle C = \frac{\pi}{3}$ (or 60°).

Note that there are faster ways to finish the solution if you remember how to factor $a^3 + b^3$.

Part D. More geometric problems

1. First step is to use Law of Sines to find ∠Y, which yields two possible angles, leading to two possible triangles. Then find ∠Z, then use Law of Sines again to get z.

One triangle: ∠Y ≈ 31.39°, ∠Z ≈ 129.61°, $z \approx 11.83$.

Other triangle: ∠Y ≈ 148.61°, ∠Z ≈ 12.39°, $z \approx 3.30$.

Triangle sketches should have reasonable angle sizes, e.g., obtuse angles should be drawn so.

2. Draw one diagonal of the quadrilateral and find its length using Law of Cosines.
Then find areas of two triangles (easiest way is to use the SAS formula) and add them.

If you drew diagonal AC: AC ≈ 9.17, area ΔABC ≈ 34.64, ∠BCA ≈ 70.89°, ∠ACD ≈ 64.11°, area ΔACD ≈ 24.74.

If you drew diagonal BD: BD ≈ 12.96, area ΔBCD ≈ 16.97, ∠DBC ≈ 19.11°, ∠ABD ≈ 40.89°, area ΔABD ≈ 42.41.

Adding the triangle areas from either approach, area of the quadrilateral is ≈ 59.38.